Note:

CHAPTER 1: BASICSSNC

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This chapter is part of the textbook:

"Fundamentals of Die Casting Design"

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GENICK BAR-MEIR, PH.D.
MINNEAPOLIS, MINNESOTA
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Fluid Mechanics	alpha		0.1.8	✔	15,000
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Open Channel Flow	NSY		0.0.0	✖	-
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NSY = Not Started Yet

CHAPTER 1

Introduction

In the recent years, many die casting companies have gone bankrupt (Doehler–Jarvis and Shelby to name a few) and many other die casting companies have been sold (St. Paul Metalcraft, Tool Products, OMC etc.). What is/are the reason/s for this situation? Some blame poor management. Others blame bad customers (which is mostly the automobile industry). Perhaps there is something to these claims. Nevertheless, one can see that the underlying reasons are the missing knowledge of how to calculate if there are profits for a production line and how to design, so that costs will be minimized. To demonstrate how the absurd situation is the fact that there is not even one company today that can calculate the actual price of any product that they are producing. Moreover, if a company is able to produce a specific product, no one in that company looks at the redesign (mold or process) in order to reduce the cost systematically.

In order to compete with other industries and other companies, the die casting industry **must** reduce the cost as much as possible (20% to 40%) and lead time significantly (by 1/2 or more). To achieve these goals, the engineer must learn to connect mold design to the cost of production (charged to the customer) and to use the correct scientific principals involved in the die casting process to reduce/eliminate the guess work. This book is part of the revolution in die casting by which science is replacing the black art of design. For the first time, a link between the cost and the design is spelled out. Many new concepts, based on scientific principles, are introduced. The old models, which was plagued by the die casting industry for many decades, are analyzed, their errors are explained and the old models are superseded.

"Science is good, but it is not useful in the floor of our plant!!" George Reed, the former president of SDCE, in 1999 announced in a meeting in the local chapter (16) of NADCA. He does not believe that there is A relationship between "science" and what he does with the die casting machine. He said that because he does not follow NADCA recommendations, he achieves good castings. For instance, he stated that the common and NADCA supported, recommendation in order to increase the gate velocity, plunger diameter needs to be decreased. He said that because he does not follow this recommendation, and/or others, that is the reason his succeeds in obtaining good castings. He is right and wrong. He is right not to follow the NADCA recommendations since they violate many basic scientific principles. One should expect that models violating scientific principles would produce unrealistic results. When such results occur, this should actually strengthen the idea that science has validity. The fact that models which appear in books today are violating scientific principals and therefore do not work should actually convince him, and others, that science does have validity. Mr. Reed is right (in certain ranges) to increase the diameter in order to increase the gate velocity as will be covered in Chapter 7.

The above example is but one of many of models that are errant and in need of correction. To this date, the author has not found so much as a single "commonly" used model that has been correct in its conclusions, trends, and/or assumptions. The wrong models/methods that have plagued the industry are: 1) critical slow plunger velocity, 2) pQ^2 diagram, 3) plunger diameter calculations, 4) runner system design, 5) vent system design, etc These incorrect models are the reasons that "science" does not work. The models presented in this book are here for the purpose of answering the questions of design in a scientific manner which will result in reduction of costs and increased product quality.

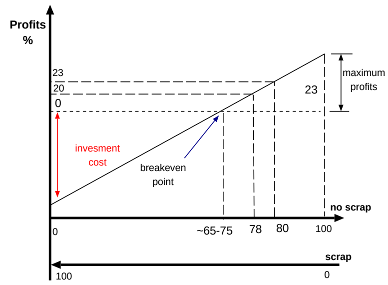

Fig. -1.1. The profits as a function of the amount of the scrap

Once the reasons to why "science" does not work are clear, one should learn the correct models for improving quality, reducing lead time and reducing production cost. The main underlying reason people are in the die casting business is to make money. One has to use science to examine what the components of production cost/scrap are and how to minimize or eliminate each of them to increase profitability. The underlying purpose of this book is to help the die caster to achieve this target.

1.1 The importance of reducing production costs

Contrary to popular belief, a reduction of a few percentage points of the production cost/scrap does not translate into the same percentage of increase in profits. The increase is a little bit more complicated function. To study the relationship further, see Figure **??** where profits are plotted as a function of the scrap. A linear function describes the relationship, when the secondary operations are neglected. The maximum loss occurs when all the material turned out to be scrap and it is referred to as the "investment cost". On the other hand, maximum profits occur when all the material becomes products (no scrap of any kind (see Figure 1.1). The breakeven point (BEP) has to exist somewhere between these two extremes. Typically, for the die casting industry, the breakeven point lies within the range of 55%–75% product (or 25%-35% scrap). Typical profits in the die casting industry are or should be about 20%. When the profits fails below 15% or typical profit in the stock exchange then the production should stop. From Figure 1.1 it can be noticed that

$$\text{relative change in profits\%} = \left(\frac{\text{new product percent} - \text{BEP}}{\text{old product percent} - \text{BEP}} - 1 \right) \times 100 \qquad (1.1)$$

Example 1.1:

What would be the effect on the profits of a small change (2%) in a amount of scrap for a job with 22% scrap (78% product) and with breakeven point of 65%?

Solution

$$\left(\frac{80 - 65}{78 - 65} - 1 \right) \times 100 = 15.3\%$$

A reduction of 2% in a amount of the scrap to be 20% (80% product) results in increase of more than 15.3% in the profits.

End solution

This is a very substantial difference. Therefore, a much bigger reduction in scrap will result in much, much bigger profits.

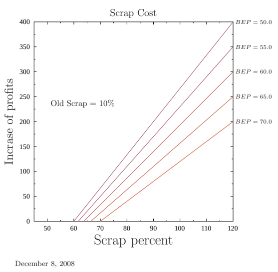

Fig a. For BEP= 10%

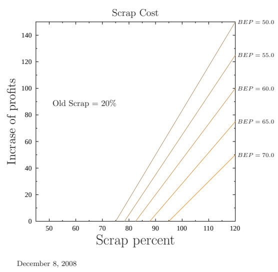

Fig b. For BEP= 10%

Fig. -1.2. The left graph depicts the increase of profits as reduction of the scrap for 10% + BEP. The right graph depicts same for for 20% + BEP.

To analysis this point further Figure intro:fig:scrapCostBEP is built for two "old" scrap values, 10% more than the BEP on the left and 20% more than BEP on the right.

The two figures (left and right) in Figure intro:fig:scrapCostBEP demonstrate that The higher BEP the change in reduction of scrap is more important. The lower the old scrap point is the more important the reduction is.

1.2 Designed/Undesigned Scrap/Cost

There can be many definitions of scrap. The best definition suited to the die casting industry should be defined as all the metal that did not become a product. There are two kinds of scrap/cost: 1) those that can be eliminated, and 2) those that can only be minimized. The first kind is referred to here as the undesigned scrap and the second is referred as designed scrap. What is the difference? It is desired not to have rejection of any part (the rejection should be zero) and of course it is not designed. Therefore, this is the undesigned scrap/cost. However, it is impossible to eliminate the runner completely and it is desirable to minimize its size in such a way that the cost will be minimized. This minimization of cost and this minimum scrap is the designed scrap/cost. The die casting engineer must distinguish between these two scrap components in order to be able to determine what should be done and what cannot be done.

Science can make a significant difference; for example, it is possible to calculate the critical slow plunger velocity and thereby eliminating (almost) air entrainment in the shot sleeve in order to minimize the air porosity. This means that air porosity will be reduced and marginal products (even poor products in some cases) are converted into good quality products. In this way, the undesigned scrap can be eliminated or minimized. Additional way of minimizing the scrap is changing several parameters. The minimum scrap/cost can be achieved when a combination of the smallest runner volume and the cheapest die casting machine are selected for a single cavity. Similar analysis can be done for multiply cavity molds. This topic will be studied further in Chapter 12.

The possibility that a parameter, which reduces the designed scrap/cost will, at the same time, reduce the undesigned scrap/cost. An example of such a parameter is the venting system design. It will be shown that there is a critical design below which air/gas is exhausted easily and above which air is trapped. In the later case, the air/gas pressure builds up and results in a poor casting (large amount of porosity) The meaning of the critical design and above and below critical design will be presented in Chapter 9. The analysis of the vent system demonstrates that a design much above the critical design and design just above the critical design yielding has almost the same results– small amount of air entrainment. One can design the vent just above the critical design so the design scrap/cost is reduced to a minimum amount possible. Now both targets have been achieved: less rejections (undesigned scrap) and less vent system volume (designed scrap). It also possible to have an opposite case in which reduction of designed scrap results in poor design. The engineer has to be aware of these points.

1.3 Linking the Production Cost to the Product Design

It is sound accounting practice to tie the cost of every aspect of production to the cost to be charged to the customer. Unfortunately, the practice today is such that the price of the products are determined by some kind of average based on the part weight plus geometry and not on the actual design and production costs. Furthermore, this idea is also perpetuated by researchers who do not have any design factor [14]. Here it is advocated to price according to the actual design and production costs. It is believed that better pricing results from such a practice. In today's practice, even after the project is finished, no one calculates the actual cost of production, let alone calculating the actual profits. The consequences of such a practice are clear: it results in no push for better design and with no idea which jobs make profits and which do not. Furthermore, considerable financial cost is incurred which could easily be eliminated. Several chapters in this book are dedicated to linking the design to the cost (end-price).

1.4 Historical Background

Die casting is, relatively speaking a very forgiving process, in which after tinkering with the several variables one can obtain a medium quality casting. For this reason there has not been any real push toward doing good research. Hence, all the major advances in the understanding of the die casting process were not sponsored by any of die casting institutes/associations. Many of the people in important positions in the die casting industry suffer from what is known as the "Detroit attitude", which is very difficult to change. "*We are making a lot of money so why change? and if do not the Government will pay for it.*". Moreover, the controlling personnel on the research funds believe that the die casting is a metallurgical manufacturing process and therefore, the research has to be carried out by either Metallurgical Engineers or Industrial Engineers. Furthermore, should come as no surprise – that people–in–charge of the research funding fund their own research. One cannot wonder if there is a relationship between so many erroneous models which have been produced and the personnel controlling the research funding. A highlight of the major points of the progress of the understanding is described herein.

The vent system design requirements were studied by some researchers, for example Suchs, Veinik, and Draper and others. These models, however, are unrealistic and do not provide no relation to the physics or realistic picture of the real requirements or of the physical situation since they ignore the major point, the air compressibility. However this research extremely poor, it highlights the idea that venting design is a must.

One of the secrets of the black art of design was that there is a range of gate velocity which creates good castings depending on the alloy properties being casted. The existence of a minimum velocity hints that a significant change in the liquid metal flow pattern occurs. Veinik linked the gate velocity to the flow pattern (atomization) and provide a qualitative physical explanation for this occurrence. Experimental work [25] showed that liquid metals, like other liquids, flow in three main patterns: a continuous flow jet, a coarse particle jet, and an atomized particle jet. Other researchers utilized the water analogy method to study flow inside the cavity for example, [6]. At present, the (minimum) required gate velocity is supported by experimental evidence which is related to the flow patterns. However, the numerical value is unknown because the experiments were poorly conducted for example, [30] the differential equations that have been "solved" are not typical to die casting. Discussion about this poor research is presented in Chapter 3. At this stage, this question is not understood.

In the late 70's, an Australian group [12] suggested adopting the pQ^2 diagram for die casting in order to calculate the gate velocity, the gate area, and other parameters. As with all the previous models they missed the major points of the calculations. As will be shown in Chapter 7, the Australian's model produce incorrect results and predict trends opposite to reality. This model took root in die casting industry for the last 25 years. Yet, one can only wonder why this well established method (the supply and demand theory which was build by Fanno (the brother of other famous Fanno from Fanno flow), which was introduced into fluid mechanics in the early of this century, reached the die casting only in the late 70's and was then erroneously implemented. This methods now properly build for the first time for the die casting industry in this book.

Until the 1980's there was no model that assisted the understanding air entrapment in the shot sleeve. Garber described the hydraulic jump in the shot sleeve and called it the "wave", probably because he was not familiar with this research area. He also developed the erroneous model which took root in the industry in spite the fact that **it never works**. One can only wonder why any die casting institutes/associations have not published this fact. Moreover, NADCA and other institutes continue to funnel large sums of money to the researchers (for example, Brevick from Ohio State) who used Garber's model even after they knew that Garber's model was totally wrong.

The turning point of the understanding was when Prof. Eckert, the father of modern heat transfer, introduced the dimensional analysis applied to the die casting process. This established a scientific approach which provided an uniform schemata for uniting experimental work with the actual situations in the die casting process. Dimensional analysis demonstrates that the fluid mechanics processes, such as filling of the cavity with liquid metal and evacuation/extraction of the air from the mold, can be dealt when the heat transfer is assumed to be negligible. However, the fluid mechanics has to be taken into account in the calculations of the heat transfer process (the solidification process).

This proved an excellent opportunity for "simple" models to predict the many parameters in the die casting process, which will be discussed later in this book. Here, two examples of new ideas that mushroomed in the inspiration of prof. Eckert's work. It has been shown that [5] the net effect of the reactions is negligible. This fact is contradictory to what was believed at that stage. The development of the critical vent area concept provided the major guidance for 1) the designs to the venting system, and 2) criterion when the vacuum system needs to be used. In this book, many of the new concepts and models, such as economy of the runner design, plunger diameter calculations, minimum runner design, etc, are described for the first time.

1.5 Numerical Simulations

Numerical simulations have been found to be very useful in many areas which lead many researchers attempting to implement them into die casting process. Considerable research work has been carried out on the problem of solidification including fluid flow which is known also as Stefan problems [21]. Minaie et al in one of the pioneered work use this knowledge and simulated the filling and the solidification of the cavity using finite difference method. Hu et al used the finite element method to improve the grid problem and to account for atomization of the liquid metal. The atomization model in the last model was based on the mass transfer coefficient. This model atomization is not appropriate. Clearly, this model is in waiting to be replaced by a realistic model to describe the mass transfer[1]. The Enthalpy method was further exploded by Swaminathan and Voller and others to study the filling and solidification problem.

While numerical simulation looks very promising, all the methods (finite difference, finite elements, or boundary elements etc) [2] suffer from several major drawbacks which prevents them from yielding reasonable results.

- There is no theory (model) that explains the heat transfer between the mold walls and the liquid metal. The lubricant sprayed on the mold change the characteristic of the heat transfer. The difference in the density between the liquid phase and solid phase creates a gap during the solidification process between the mold and the ingate which depends on the geometry. For example, Osborne et al showed that a commercial software (MAGMA) required fiddling with the heat transfer coefficient to get the numerical simulation match the experimental results[3].

- As it was mentioned earlier, it is not clear when the liquid metal flows as a spray and when it flows as continuous liquid. Experimental work has demonstrated that the flow, for a large part of the filling time, is atomized [4].

- The pressure in the mold cavity in all the commercial codes are calculated without taking into account the resistance to the air flow out. Thus, built–up pressure in the cavity is poorly estimated, or even not realistic, and therefore the characteristic flow of the liquid metal in the mold cavity is poorly estimated as well.

- The flow in all the simulations is assumed to be turbulent flow. However, time and space are required to achieved a fully turbulent flow. For example, if the flow at the entrance to a pipe with the typical conditions in die casting is laminar (actually it is a plug flow) it will take a runner with a length of about 10[m] to achieved fully developed flow. With this in mind, clearly some part of the flow is laminar. Additionally, the solidification process is faster compared to the dissipation process in the initial stage, so it is also a factor in changing the flow from a turbulent (in case the flow is turbulent) to a laminar flow.

[1] One finds that it is the easiest to critic one's own work or where he/she was involved.

[2] Commercial or academic versions.

[3] Actually, they attempted to prove that the software is working very well. However, the fact that coefficient need to field is excellent proof why this work is meaningless.

- The liquid metal velocity at the entrance to the runner is assumed for the numerical simulation and not calculated. In reality this velocity has to be calculated utilizing the pQ^2 diagram.

- If turbulence exists in the flow field, what is the model that describes it adequately? Clearly, model such $k - \epsilon$ are based on isentropic homogeneous with mild change in the properties cannot describe situations where the flow changes into two-phase flow (solid-liquid flow) etc.

- The heat extracted from the die is done by cooling liquid (oil or water). In most models (all the commercial models) the mechanism is assumed to be by "regular cooling". In actuality, some part of the heat is removed by boiling heat transfer.

- The governing equations in all the numerical models, that I am aware of, neglect the dissipation term in during the solidification. The dissipation term is the most important term in that case.

One wonders how, with unknown flow pattern (or correct flow pattern), unrealistic pressure in the mold, wrong heat removal mechanism (cooling method), erroneous governing equation in the solidification phase, and inappropriate heat transfer coefficient, a simulation could produce any realistic results. Clearly, much work is need to be done in these areas before any realistic results should be expected from any numerical simulation. Furthermore, to demonstrate this point, there are numerical studies that assume that the flow is turbulent, continuous, no air exist (or no air leaving the cavity) and proves with their experiments that their model simulate "reality" [23]. On the other hand, other numerical studies assumed that the flow does not have any effect on the solidification and of course have their experiments to support this claim [11]. Clearly, this contradiction suggest several options:

- Both of the them are right and the model itself does not matter.

- One is right and the other one is wrong.

- Both of them are wrong.

The third research we mentioned here is an example where the calculations can be shown to be totally wrong and yet the researchers have experimental proofs to back them up. Viswanathan et al studied a noble process in which the liquid metal is poured into the cavity and direct pressure is applied to the cavity. In their calculations the authors assumed that metal enter to the cavity and fill the whole entrance (gate) to the cavity. Based on this assumption their model predict defects in certain geometry. A critical examination of this model present the following. The assumption of no air flow out by the authors (was "explained" privately that air amount is a small and therefore not important) is very critical as will be shown here. The volumetric air flow rate into the cavity has to be on average equal to liquid metal flow rate (conservation of volume for constant density). Hence, air velocity has to be approximately infinite to achieve zero vent area. Conversely, if the assumption that the air flows in the same velocity as the liquid entering the cavity, liquid metal flow area is a half what is assume in the researchers model. In realty, the flow of the liquid metal is in the two phase region and in this case, it is like turning a bottle full of water over and liquid inside flows as "blobs" [4]. More information can be found on reversible flow in this author book in Potto series of "Basics of Fluid Mechanics." In this case the whole calculations do not have much to do with reality since the velocity is not continuous and different from what was calculated.

Another example of such study is the model of the flow in the shot sleeve by Backer and Sant from EKK [2][5]. The researchers assumed that the flow is turbulent and they justified it because they calculated and found a "jet" with extreme velocity. Unfortunately, all the experimental evidence demonstrate that there is no such jet [24]. It seems that this jet results from the "poor" boundary and initial conditions[6]. In the presentation, the researchers also stated that results they obtained for laminar and turbulent flow were the same[7] while a simple analysis can demonstrate the difference is very large. Also, one can wonder how liquid with zero velocity to be turbulent. With these results one can wonder if the code is of any value or the implementation is at fault.

The bizarre belief that the numerical simulations are a panacea to all the design problem is very popular in the die casting industry. Any model has to describe and account for the physical situation in order to be useful. Experimental evidence which is supporting wrong models as a real evidence is nonsense. Clearly some wrong must be there. For example, see the paper by Murray and colleague in which they use the fact that two unknown companies (somewhere in the outer space maybe?) were using their model to claim that it is correct.. A proper way can be done by numerical calculations based on real physics principles which produce realistic results. Until that point come, the reader should be suspicious about any numerical model and its supporting evidence.[8]

1.6 "Integral" Models

Unfortunately, the numerical simulations of the liquid metal flow and solidification process do not yield reasonable results at the present time. This problem has left the die casting engineers with the usage of the "integral approach" method. In this method the calculations are broken into simplified models. One of the most important tool in this approach is the pQ^2 diagram, one of the manifestations of the supply and demand theory. In this diagram, an engineer insures that die casting machine ability can fulfill the die mold design requirements; the liquid metal is injected at the right velocity range

[4]Try it your self! fill a bottle and turn it upside and see what happens.

[5]It was suggested by several people that the paper was commissioned by NADCA to counter Bar-Meir's equation to shot sleeve. This fact is up to the reader to decide if it is correct.

[6]The boundary and initial conditions were not spelled out in the paper!! However they were implicitly stated in the presentation.

[7]So why to use the complicate turbulent model?

[8]With all these harsh words, I would like to take the opportunity for the record, I do think that work by Davey's group is a good one. They have inserted more physics (for example the boiling heat transfer) into their models which I hope in the future, leads us to have realistic numerical models.

and the filling time is small enough to prevent premature freezing. One can, with the help of the pQ^2 diagram and by utilizing experimental values for desired filling time and gate velocities improve the quality of the casting. The gate velocity has to be above a certain value to assure atomization and below a critical value to prevent erosion of the mold. This two values are experimental and no reliable theory is available today known. The correct model for the pQ^2 diagram has been developed and will be presented in Chapter 7. A by–product of the above model is the plunger diameter calculations and it is discussed in Chapter 7.

It turned out that many of the design parameters in die casting have a critical point above which good castings are produced and below which poor castings are produced. Furthermore, **much** above and **just** above the critical point do not change much the quality but costs much more. This fact is where the economical concepts plays a significant role. Using these concepts, one can increase the profitability significantly, and obtain very good quality casting and reduce the leading time. Additionally, the main cost components like machine cost and other are analyzed which have to be taken into considerations when one chooses to design the process will be discussed in the Chapter 12.

Porosity can be divided into two main categories; shrinkage porosity and gas/air entrainment. The porosity due to entrapped gases constitutes a large part of the total porosity. The creation of gas/air entrainment can be attributed to at least four categories: lubricant evaporation (and reaction processes[9]), vent locations (last place to be filled), mixing processes, and vent/gate area. The effects of lubricant evaporation have been found to be insignificant. The vent location(s) can be considered partially solved since only qualitative explanation exist. The mixing mechanisms are divided into two zones: the mold, and the shot sleeve. Some mixing processes have been investigated and can be considered solved. The requirement on the vent/gate areas is discussed in Chapter 9. When the mixing processes are very significant in the mold, other methods are used and they include: evacuating the cavities (vacuum venting), Pore Free Technique (in zinc and aluminum casting) and squeeze casting. The first two techniques are used to extract the gases/air from the shot sleeve and die cavity before the gases have the opportunity to mix with the liquid metal. The squeeze casting is used to increase the capillary forces and therefore, to minimize the mixing processes. All these solutions are cumbersome and more expensive and should be avoided if possible.

The mixing processes in the runners, where the liquid metal flows vertically against gravity in relatively large conduit, are considered to be insignificant[10]. The enhanced air entrainment in the shot sleeve is attributed to operational conditions for which a blockage of the gate by a liquid metal wave occurs before the air is exhausted. Consequently, the residual air is forced to be mixed into the liquid metal in the shot sleeve. With Bar-Meir's formula, one can calculate the correct critical slow plunger velocity and this will be discussed in Chapter 8.

[9]Some researchers view the chemical reactions (e.g. release of nitrogen during solidification process) as category by itself.

[10]Some work has been carried out and hopefully will be published soon. And inside, in the book "Basic of Fluid Mechanics" in the two phase chapter some inside was developed.

1.7 Summary

It is an exciting time in the die casing industry because for the first time, an engineer can start using real science in designing the runner/mold and the die casting process. Many new models have been build and many old techniques mistake have been removed. It is the new revolution in the die casting industry.

CHAPTER 2

Basic Fluid Mechanics

2.1 Introduction

This chapter is presented to fill the void in basic fluid mechanics to the die casting community. It was observed that knowledge in this area cannot be avoided. The design of the process as well as the properties of casting (especially magnesium alloys) are determined by the fluid mechanics/heat transfer processes. It is hoped that others will join to spread this knowledge. There are numerous books for introductory fluid mechanics but the Potto series book "Basic of Fluid Mechanics" is a good place to start. This chapter is a summary of that book plus some pieces from the "Fundamentals of Compressible Flow Mechanics." It is hoped that the reader will find this chapter interesting and will further continue expanding his knowledge by reading the full Potto books on fluid mechanics and compressible flow.

First we will introduce the nature of fluids and basic concepts from thermodynamics. Later the integral analysis will be discussed in which it will be divided into introduction of the control volume concept and Continuity equations. The energy equation will be explained in the next section. Later, the momentum equation will be discussed. Lastly, the chapter will be dealing with the compressible flow gases. Here it will be refrained from dealing with topics such boundary layers, non–viscous flow, machinery flow etc which are not essential to understand the rest of this book. Nevertheless, they are important and it is advisable that the reader will read on these topics as well.

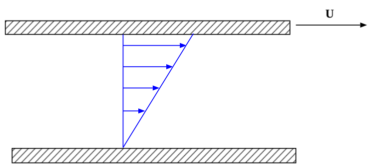

Fig. -2.1. The velocity distribution in Couette flow

2.2 What is fluid? Shear stress

Fluid, in this book, is considered as a substance that "moves" continuously and permanently when exposed to a shear stress. The liquid metals are an example of such substance. However, the liquid metals do not have to be in the liquidus phase to be considered liquid. Aluminum at approximately 400^0C is continuously deformed when shear stress are applied. The whole semi–solid die casting area deals with materials that "looks" solid but behaves as liquid.

2.2.1 What is Fluid?

The fluid is mainly divided into two categories: liquids and gases. The main difference between the liquids and gases state is that gas will occupy the whole volume while liquids has an almost fixed volume. This difference can be, for most practical purposes considered, sharp even though in reality this difference isn't sharp. The difference between a gas phase to a liquid phase above the critical point are practically minor. But below the critical point, the change of water pressure by 1000% only change the volume by less than 1 percent. For example, a change in the volume by more than 5% will require tens of thousands percent change of the pressure. So, if the change of pressure is significantly less than that, then the change of volume is at best 5%. Hence, the pressure will not affect the volume. In gaseous phase, any change in pressure directly affects the volume. The gas fills the volume and liquid cannot. Gas has no free interface/surface (since it does fill the entire volume).

2.2.2 What is Shear Stress?

The shear stress is part of the pressure tensor. However, here it will be treated as a separate issue. In solid mechanics, the shear stress is considered as the ratio of the force acting on area in the direction of the forces perpendicular to area. Different from solid, fluid cannot pull directly but through a solid surface. Consider liquid that undergoes a shear stress between a short distance of two plates as shown in Figure (**??**).

The upper plate velocity generally will be

$$U = f(A, F, h) \tag{2.1}$$

Where A is the area, the F denotes the force, h is the distance between the plates. From solid mechanics study, it was shown that when the force per area increases, the velocity of the plate increases also. Experiments show that the increase of height will increase the velocity up to a certain range. Consider moving the plate with a zero lubricant ($h \sim 0$) (results in large force) or a large amount of lubricant (smaller force). In this discussion, the aim is to develop differential equation, thus the small distance analysis is applicable.

For cases where the dependency is linear, the following can be written

$$U \propto \frac{h\,F}{A} \tag{2.2}$$

Equations (2.2) can be rearranged to be

$$\frac{U}{h} \propto \frac{F}{A} \tag{2.3}$$

Shear stress was defined as

$$\tau_{xy} = \frac{F}{A} \tag{2.4}$$

From equations (2.3) and (2.4) it follows that ratio of the velocity to height is proportional to shear stress. Hence, applying the coefficient to obtain a new equality as

$$\tau_{xy} = \mu \, \frac{U}{h} \tag{2.5}$$

Where μ is called the absolute viscosity or dynamic viscosity.

In steady state, the distance the upper plate moves after small amount of time, δt is

$$d\ell = U\,\delta t \tag{2.6}$$

From figure (2.2) it can be noticed that for a small angle, the regular approximation provides

$$d\ell = U\,\delta t = \overbrace{h\,\delta\beta}^{geometry} \tag{2.7}$$

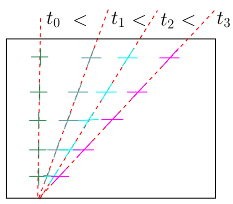

Fig. -2.2. The deformation of fluid due to shear stress as progression of time.

From equation (2.7) it follows that

$$U = h\frac{\delta\beta}{\delta t} \tag{2.8}$$

Combining equation (2.8) with equation (2.5) yields

$$\tau_{xy} = \mu \, \frac{\delta\beta}{\delta t} \tag{2.9}$$

If the velocity profile is linear between the plate (it will be shown later that it is consistent with derivations of velocity), then it can be written for small angle that

$$\frac{\delta\beta}{\delta t} = \frac{dU}{dy} \tag{2.10}$$

Materials which obey equation (2.9) are referred to as Newtonian fluid.

For liquid metal used in the die casting industry, this property should be considered as Newtonian fluid.

2.3 Thermodynamics and mechanics concepts

2.3.1 Thermodynamics

In this section, a review of several definitions of common thermodynamics terms is presented. This introduction is provided to bring familiarity of the material back to the student.

2.3.2 Basic Definitions

The following basic definitions are common to thermodynamics and will be used in this book.

Work

In mechanics, the work was defined as

$$\text{mechanical work} = \int \mathbf{F} \bullet \mathbf{d}\ell = \int P dV \tag{2.11}$$

This definition can be expanded to include two issues. The first issue that must be addressed, that work done on the surroundings by the system boundaries similarly is positive. Two, there is a transfer of energy so that its effect can cause work. It must be noted that electrical current is a work while heat transfer isn't.

System

This term will be used in this book and it is defined as a continuous (at least partially) fixed quantity of matter (neglecting Einstein's law effects). For almost all engineering purposes this law is reduced to two separate laws: mass conservation and energy conservation. Our system can receive energy, work, etc as long as the mass remains constant the definition is not broken.

Thermodynamics First Law

This law refers to conservation of energy in a non accelerating system. Since all the systems can be calculated in a non accelerating system, the conservation is applied to all systems. The statement describing the law is the following:

$$Q_{12} - W_{12} = E_2 - E_1 \tag{2.12}$$

The system energy is a state property. From the first law it directly implies that for process without heat transfer (adiabatic process) the following is true

$$W_{12} = E_1 - E_2 \tag{2.13}$$

Interesting results of equation (2.13) is that the way the work is done and/or intermediate states are irrelevant to final results. The internal energy is the energy that depends on the other properties of the system. Example: for pure/homogeneous and simple gases it depends on two properties like temperature and pressure. The internal energy is denoted in this book as E_U and it will be treated as a state property.

The system potential energy is dependent upon the body force. A common body force is gravity. For such body force, the potential energy is mgz where g is the gravity force (acceleration), m is the mass and the z is the vertical height from a datum. The kinetic energy is

$$K.E. = \frac{mU^2}{2} \tag{2.14}$$

Thus the energy equation can be written as

$$\frac{m{U_1}^2}{2} + \overbrace{m\,g\,z_1}^{B_f} + E_{U\,1} + Q = \frac{m{U_2}^2}{2} + \overbrace{m\,g\,z_2}^{B_f} + E_{U\,2} + W \tag{2.15}$$

where B_f is a body force. For the unit mass of the system equation (2.15) is transformed into

$$\frac{{U_1}^2}{2} + gz_1 + E_{u1} + q = \frac{{U_2}^2}{2} + gz_2 + E_{u2} + w \tag{2.16}$$

where q is the energy per unit mass and w is the work per unit mass. The "new" internal energy, E_u, is the internal energy per unit mass.

Since the above equations are true between arbitrary points, choosing any point in time will make it correct. Thus, differentiating the energy equation with respect to time yields the rate of change energy equation. The rate of change of the energy transfer is

$$\frac{DQ}{Dt} = \dot{Q} \tag{2.17}$$

In the same manner, the work change rate transferred through the boundaries of the system is

$$\frac{DW}{Dt} = \dot{W} \tag{2.18}$$

Since the system is with a fixed mass, the rate energy equation is

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + mU\frac{DU}{Dt} + m\frac{D\,g\,z}{Dt} \tag{2.19}$$

For the case were the body force, $B_f = g$, is constant with time like in the case of gravity equation (2.19) reduced to

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + mU\frac{DU}{Dt} + m\,g\frac{D\,z}{Dt} \tag{2.20}$$

The time derivative operator, D/Dt is used instead of the common notation because it refers to system property derivative.

Thermodynamics Second Law

There are several definitions of the second law. No matter which definition is used to describe the second law it will end in a mathematical form. The most common mathematical form is Clausius inequality which state that

$$\oint \frac{\delta Q}{T} \geq 0 \tag{2.21}$$

The integration symbol with the circle represent integral of cycle (therefore circle) of system which returns to the same condition. If there is no lost, it is referred as a reversible process and the inequality change to equality.

$$\oint \frac{\delta Q}{T} = 0 \tag{2.22}$$

The last integral can go though several states. These states are independent of the path the system goes through. Hence, the integral is independent of the path. This observation leads to the definition of entropy and designated as S and the derivative of entropy is

$$ds \equiv \left(\frac{\delta Q}{T}\right)_{\text{rev}} \tag{2.23}$$

Performing integration between two states results in

$$S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T}\right)_{\text{rev}} = \int_1^2 dS \tag{2.24}$$

One of the conclusions that can be drawn from this analysis is for reversible and adiabatic process $dS = 0$. Thus, the process in which it is reversible and adiabatic, the entropy remains constant and referred to as isentropic process. It can be noted that there is a possibility that a process can be irreversible and the right amount of heat transfer to have zero change entropy change. Thus, the reverse conclusion that zero change of entropy leads to reversible process, isn't correct.

For reversible process equation (2.22) can be written as

$$\delta Q = T dS \tag{2.25}$$

and the work that the system is doing on the surroundings is

$$\delta W = P dV \tag{2.26}$$

Substituting equations (2.25) (2.26) into (2.20) results in

$$T dS = d\,E_U + P dV \tag{2.27}$$

Even though the derivation of the above equations were done assuming that there is no change of kinetic or potential energy, it still remains valid for all situations. Furthermore, it can be shown that it is valid for reversible and irreversible processes.

Enthalpy

It is a common practice to define a new property, which is the combination of already defined properties, the enthalpy of the system.

$$H = E_U + PV \tag{2.28}$$

The specific enthalpy is enthalpy per unit mass and denoted as, h.

Or in a differential form as

$$dH = dE_U + dP\,V + P\,dV \tag{2.29}$$

Combining equations (2.28) the (2.27) yields

$$\boxed{TdS = dH - V dP} \tag{2.30}$$

For isentropic process, equation (2.27) is reduced to $dH = VdP$. The equation (2.27) in mass unit is

$$Tds = du + Pdv = dh - \frac{dP}{\rho} \tag{2.31}$$

when the density enters through the relationship of $\rho = 1/v$.

Specific Heats

The change of internal energy and enthalpy requires new definitions. The first change of the internal energy and it is defined as the following

$$\boxed{C_v \equiv \left(\frac{\partial E_u}{\partial T}\right)} \tag{2.32}$$

And since the change of the enthalpy involve some kind of work, it is defined as

$$\boxed{C_p \equiv \left(\frac{\partial h}{\partial T}\right)} \tag{2.33}$$

The ratio between the specific pressure heat and the specific volume heat is called the ratio of the specific heats and it is denoted as, k.

$$\boxed{k \equiv \frac{C_p}{C_v}} \tag{2.34}$$

For liquid metal used in die casting, the ratio of the specific heats is bite higher than one (1) and therefore the difference between them is almost zero and therefore referred as C.

Equation of state

Equation of state is a relation between state variables. Normally the relationship of temperature, pressure, and specific volume define the equation of state for gases. The simplest equation of state referred to as ideal gas and it is defined as

$$P = \rho R T \tag{2.35}$$

Application of Avogadro's law, that "all gases at the same pressures and temperatures have the same number of molecules per unit of volume," allows the calculation of a "universal gas constant." This constant to match the standard units results in

$$\bar{R} = 8.3145 \frac{kj}{kmol\ K} \tag{2.36}$$

Thus, the specific gas can be calculated as

$$R = \frac{\bar{R}}{M} \tag{2.37}$$

The specific constants for select gas at 300K is provided in table 2.1.

From equation (2.35) of state for perfect gas it follows

$$d(Pv) = RdT \tag{2.38}$$

For perfect gas

$$dh = dE_u + d(Pv) = dE_u + d(RT) = f(T) \text{ (only)} \tag{2.39}$$

From the definition of enthalpy it follows that

$$d(Pv) = dh - dE_u \tag{2.40}$$

Utilizing equation (2.38) and substituting into equation (2.40) and dividing by dT yields

$$C_p - C_v = R \tag{2.41}$$

This relationship is valid only for ideal/perfect gases.

The ratio of the specific heats can be expressed in several forms as

$$C_v = \frac{R}{k-1} \tag{2.42}$$

Table -2.1. Properties of Various Ideal Gases [300K]

Gas	Chemical Formula	Molecular Weight	R $\left[\frac{kj}{KgK}\right]$	C_v $\left[\frac{kj}{KgK}\right]$	C_P $\left[\frac{kj}{KgK}\right]$	k
Air	-	28.970	0.28700	1.0035	0.7165	1.400
Argon	Ar	39.948	0.20813	0.5203	0.3122	1.400
Butane	C_4H_{10}	58.124	0.14304	1.7164	1.5734	1.091
Carbon Dioxide	CO_2	44.01	0.18892	0.8418	0.6529	1.289
Carbon Monoxide	CO	28.01	0.29683	1.0413	0.7445	1.400
Ethane	C_2H_6	30.07	0.27650	1.7662	1.4897	1.186
Ethylene	C_2H_4	28.054	0.29637	1.5482	1.2518	1.237
Helium	He	4.003	2.07703	5.1926	3.1156	1.667
Hydrogen	H_2	2.016	4.12418	14.2091	10.0849	1.409
Methane	CH_4	16.04	0.51835	2.2537	1.7354	1.299
Neon	Ne	20.183	0.41195	1.0299	0.6179	1.667
Nitrogen	N_2	28.013	0.29680	1.0416	0.7448	1.400
Octane	C_8H_{18}	114.230	0.07279	1.7113	1.6385	1.044
Oxygen	O_2	31.999	0.25983	0.9216	0.6618	1.393
Propane	C_3H_8	44.097	0.18855	1.6794	1.4909	1.327
Steam	H_2O	18.015	0.48152	1.8723	1.4108	1.327

$$C_p = \frac{k\,R}{k-1} \tag{2.43}$$

The specific heats ratio, k value ranges from unity to about 1.667. These values depend on the molecular degrees of freedom (more explanation can be obtained in Van Wylen "F. of Classical thermodynamics.") The values of several gases can be approximated as ideal gas and are provided in Table (2.1).

The entropy for ideal gas can be simplified as the following

$$s_2 - s_1 = \int_1^2 \left(\frac{dh}{T} - \frac{dP}{\rho T}\right) \tag{2.44}$$

Using the identities developed so far one can find that

$$s_2 - s_1 = \int_1^2 C_p \frac{dT}{T} - \int_1^2 \frac{R\,dP}{P} = C_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} \tag{2.45}$$

Or using specific heats ratio equation (2.45) transformed into

$$\frac{s_2 - s_1}{R} = \frac{k}{k-1}\ln\frac{T_2}{T_1} - \ln\frac{P_2}{P_1} \tag{2.46}$$

For isentropic process, $\Delta s = 0$, the following is obtained

$$\ln\frac{T_2}{T_1} = \ln\left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} \tag{2.47}$$

There are several famous identities that results from equation (2.47) as

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{P_2}{P_1}\right)^{k-1} \tag{2.48}$$

The ideal gas model is a simplified version of the real behavior of real gas. The real gas has a correction factor to account for the deviations from the ideal gas model. This correction factor is referred to as the compressibility factor and defined as

$$Z = \frac{PV}{RT} \tag{2.49}$$

Control Volume

The control volume was introduced by L. Euler[1] In the control volume (c.v) the focus is on specific volume which mass can enter and leave. The simplest c.v. is when the boundaries are fixed and it is referred to as the *Non–deformable c.v.*. The conservation of mass to such system can be reasonably approximated by

$$\frac{d}{dt}\int_{V_{c.v.}} \rho dV = -\int_{S_{c.v.}} \rho V_{rn} dA \tag{2.50}$$

This equation states the change in the volume came from the difference of masses being added through the boundary.

put two examples of simple for mass conservation.

For deformable c.v.

$$\frac{d}{dt}\int_{V_{c.v.}} \rho dV = \int_{V_{c.v.}} \frac{d\rho}{dt} dV + \int_{S_{c.v.}} \rho V_{rn} dA \tag{2.51}$$

[1]A blind man known as the master of calculus, made his living by being a tutor, can you imagine he had eleven kids: where he had the time and energy to develop all the great theory and mathematics.

2.3.3 Momentum Equation

The second Newton law of motion is written mathematically as

$$\Sigma F = \frac{D}{Dt} mV \tag{2.52}$$

This explanation, of course, for fluid particles can be written as

$$\Sigma F = \frac{D}{Dt} \int_{V_{sys}} V \rho dV \tag{2.53}$$

or more explicitly it can be written as

$$\Sigma F = \frac{d}{dt} \int_{V_{c.v.}} \rho V dV + \int_{A_{c.v.}} \rho V \cdot V_{rn} dA \tag{2.54}$$

2.3.4 Compressible flow

This material is extensive and requires a semester for student to have good understanding of this complex material. Yet to give very minimal information is seems to to be essential to the understanding of the venting design. The summary material here is derived from the book "Fundamentals of Compressible Flow Mechanics."

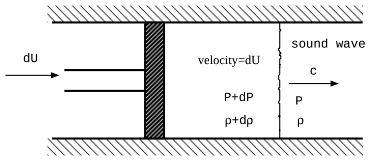

Fig. -2.3. A very slow moving piston in a still gas.

2.3.5 Speed of Sound

The speed of sound is a very important parameter in the die casting process because it effects and explains the choking in the die casting process. What is the speed of the small disturbance +as it travels in a "quiet" medium? This velocity is referred to as the speed of sound. To answer this question, consider a piston moving from the left to the right at a relatively small velocity (see Figure 2.3). The information that the piston is moving passes thorough a single "pressure pulse." It is assumed that if the velocity of the piston is infinitesimally small, the pulse will be infinitesimally small. Thus, the pressure and density can be assumed to be continuous.

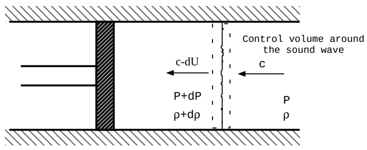

Fig. -2.4. Stationary sound wave and gas moves relative to the pulse.

It is convenient to look at a control volume which is attached to a pressure pulse. Applying the mass balance yields

$$\rho c = (\rho + d\rho)(c - dU) \tag{2.55}$$

or when the higher term $dUd\rho$ is neglected yields

$$\rho dU = cd\rho \Longrightarrow dU = \frac{cd\rho}{\rho} \tag{2.56}$$

From the energy equation (Bernoulli's equation), assuming isentropic flow and neglecting the gravity results

$$\frac{(c-dU)^2 - c^2}{2} + \frac{dP}{\rho} = 0 \tag{2.57}$$

neglecting second term (dU^2) yield

$$-cdU + \frac{dP}{\rho} = 0 \tag{2.58}$$

Substituting the expression for dU from equation (2.56) into equation (2.58) yields

$$c^2\left(\frac{d\rho}{\rho}\right) = \frac{dP}{\rho} \Longrightarrow c^2 = \frac{dP}{d\rho} \tag{2.59}$$

It is shown in the book "Fundamentals of Compressible Fluid Mechanics" that relationship between n, Z and k is

$$n = \overbrace{\frac{C_p}{C_v}}^{k}\left(\frac{z + T\left(\frac{\partial z}{\partial T}\right)_\rho}{z + T\left(\frac{\partial z}{\partial T}\right)_P}\right) \tag{2.60}$$

Note that n approaches k when $z \rightarrow 1$ and when z is constant. The speed of sound for a real gas can be obtained in similar manner as for an ideal gas

$$\frac{dP}{d\rho} = nzRT \tag{2.61}$$

Speed of Sound in Almost Incompressible Liquid

Even liquid metal *normally* is assumed to be incompressible but in reality it has a small and important compressible aspect. The ratio of the change in the fractional volume to pressure or compression is referred to as the bulk modulus of the material. The mathematical definition of bulk modulus is as follows

$$B = \rho\frac{dP}{d\rho} \tag{2.62}$$

In physical terms it can be written as

$$c = \sqrt{\frac{elastic\ property}{inertial\ property}} = \sqrt{\frac{B}{\rho}} \tag{2.63}$$

In summary, the speed of sound in liquid metals is about 5 times faster than the speed of sound in gases in the chamber.

2.3.6 Choked Flow

In this section a discussion on a steady state flow through a smooth and continuous area flow rate is presented which include the flow through a converging–diverging nozzle. The isentropic flow models are important because of two main reasons:

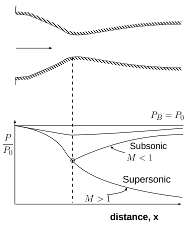

Fig. -2.5. Flow of a compressible substance (gas) through a converging–diverging nozzle.

Stagnation State for Ideal Gas Model

It is assumed that the flow is one–dimensional. Figure (2.5) describes a gas flow through a converging–diverging nozzle. It has been found that a theoretical state known as the stagnation state is very useful in which the flow is brought into a complete motionless condition in isentropic process without other forces (e.g. gravity force). Several properties can be represented by this theoretical process which include temperature, pressure, and density etc and denoted by the subscript "0."

A dimensionless velocity and it is referred as Mach number for the ratio of velocity to speed of sound as

$$M \equiv \frac{U}{c} \tag{2.64}$$

The temperature ratio reads

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \tag{2.65}$$

The ratio of stagnation pressure to the static pressure can be expressed as the function of the temperature ratio because of the isentropic relationship as

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{2.66}$$

In the same manner the relationship for the density ratio is

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} \tag{2.67}$$

A new useful definition is introduced for the case when $M = 1$ and denoted by superscript "$*$." The special case of ratio of the star values to stagnation values are dependent only on the heat ratio as the following:

$$\frac{T^*}{T_0} = \frac{{c^*}^2}{{c_0}^2} = \frac{2}{k+1} \tag{2.68}$$

and

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{2.69}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \tag{2.70}$$

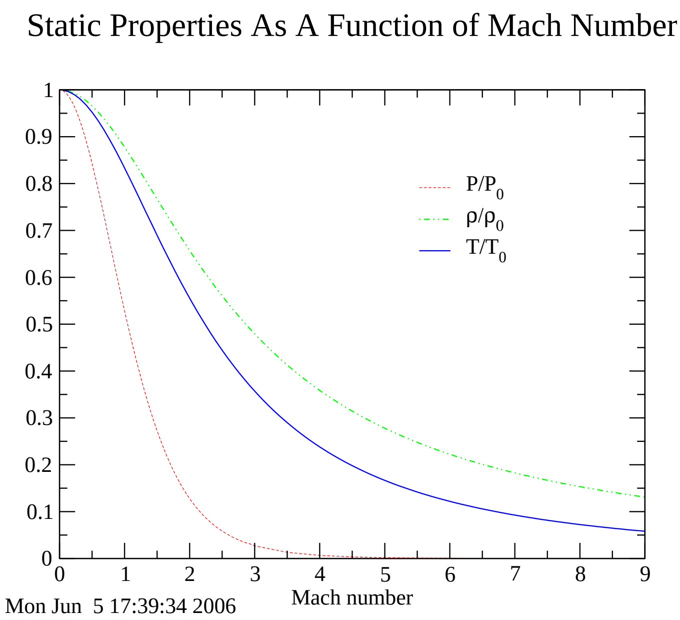

Fig. -2.6. The stagnation properties as a function of the Mach number, k=1.4

The definition of the star Mach is ratio of the velocity and star speed of sound at $M = 1$.

The flow in a converging–diverging nozzle has two models: First is isentropic and adiabatic model. Second is isentropic and isothermal model. Clearly, the stagnation temperature, T_0, is constant through the adiabatic flow because there isn't heat transfer. Therefore, the stagnation pressure is also constant through the flow because of the isentropic flow. Conversely, in mathematical terms, equation (2.65) and equation (2.66) are the same. If the right hand side is constant for one variable, it is constant for the other. In the same argument, the stagnation density is constant through the flow. Thus, knowing the Mach number or the temperature will provide all that is needed to find the other properties. The only properties that need to be connected are the cross section area and the Mach number. Examination of the relation between properties can then be carried out.

The Properties in the Adiabatic Nozzle

When there is no external work and heat transfer, the energy equation, reads

$$dh + UdU = 0 \tag{2.71}$$

Differentiation of continuity equation, $\rho AU = \dot{m} = constant$, and dividing by the continuity equation reads

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dU}{U} = 0 \tag{2.72}$$

The thermodynamic relationship between the properties can be expressed as

$$Tds = dh - \frac{dP}{\rho} \tag{2.73}$$

For isentropic process $ds \equiv 0$ and combining equations (2.71) with (2.73) yields

$$\frac{dP}{\rho} + UdU = 0 \tag{2.74}$$

Differentiation of the equation state (perfect gas), $P = \rho RT$, and dividing the results by the equation of state (ρRT) yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{2.75}$$

Obtaining an expression for dU/U from the mass balance equation (2.72) and using it in equation (2.74) reads

$$\frac{dP}{\rho} - U^2 \overbrace{\left[\frac{dA}{A} + \frac{d\rho}{\rho}\right]}^{\frac{dU}{U}} = 0 \tag{2.76}$$

Rearranging equation (2.76) so that the density, ρ, can be replaced by the static pressure, dP/ρ yields

$$\frac{dP}{\rho} = U^2 \left(\frac{dA}{A} + \frac{d\rho}{\rho}\frac{dP}{dP}\right) = U^2 \left(\frac{dA}{A} + \overbrace{\frac{d\rho}{dP}}^{\frac{1}{c^2}} \frac{dP}{\rho}\right) \tag{2.77}$$

Recalling that $dP/d\rho = c^2$ and substitute the speed of sound into equation (2.77) to obtain

$$\frac{dP}{\rho}\left[1-\left(\frac{U}{c}\right)^2\right] = U^2\frac{dA}{A} \tag{2.78}$$

Or in a dimensionless form

$$\frac{dP}{\rho}\left(1-M^2\right) = U^2\frac{dA}{A} \tag{2.79}$$

Equation (2.79) is a differential equation for the pressure as a function of the cross section area. It is convenient to rearrange equation (2.79) to obtain a variables separation form of

$$dP = \frac{\rho U^2}{A}\frac{dA}{1-M^2} \tag{2.80}$$

Before going further in the mathematical derivation it is worth while to look at the physical meaning of equation (2.80). The term $\rho U^2/A$ is always positive (because all the three terms can be only positive). Now, it can be observed that dP can be positive or negative depending on the dA and Mach number. The meaning of the sign change for the pressure differential is that the pressure can increase or decrease. It can be observed that the critical Mach number is one. If the Mach number is larger than one than dP has opposite sign of dA. If Mach number is smaller than one dP and dA have the same sign. For the subsonic branch $M < 1$ the term $1/(1-M^2)$ is positive hence

$$dA > 0 \Longrightarrow dP > 0$$
$$dA < 0 \Longrightarrow dP < 0$$

From these observations the trends are similar to those in incompressible fluid. An increase in area results in an increase of the static pressure (converting the dynamic pressure to a static pressure). Conversely, if the area decreases (as a function of x) the pressure decreases. Note that the pressure decrease is larger in compressible flow compared to incompressible flow.

For the supersonic branch $M > 1$, the phenomenon is different. For $M > 1$ the term $1/1 - M^2$ is negative and change the character of the equation.

$$dA > 0 \Rightarrow dP < 0$$
$$dA < 0 \Rightarrow dP > 0$$

This behavior is opposite to incompressible flow behavior.

For the special case of $M = 1$ (sonic flow) the value of the term $1 - M^2 = 0$ thus mathematically $dP \to \infty$ or $dA = 0$. Since physically dP can increase only in a finite amount it must be that $dA = 0$.It must also be noted that when $M = 1$ occurs only when $dA = 0$. However, the opposite, not necessarily means that when $dA = 0$ that $M = 1$. In that case, it is possible that $dM = 0$ thus the diverging side is in the subsonic branch and the flow isn't choked.

Isentropic Isothermal Flow Nozzle

In this section, the other extreme case model where the heat transfer to the gas is perfect, (e.g. Eckert number combination is very small) is presented. Again in reality the heat transfer is somewhere in between the two extremes. So, knowing the two limits provides a tool to examine where the reality should be expected. The perfect gas model is again assumed. In isothermal process the perfect gas model reads

$$P = \rho R T \leadsto dP = d\rho R T \tag{2.81}$$

Substituting equation (2.81) into the momentum equation[2] yields

$$U dU + \frac{R T dP}{P} = 0 \tag{2.82}$$

Integration of equation (2.82) yields the Bernoulli's equation for ideal gas in isothermal process which reads

$$\leadsto \frac{{U_2}^2 - {U_1}^2}{2} + R T \ln \frac{P_2}{P_1} = 0 \tag{2.83}$$

Then the stagnation velocity is

$$U = \sqrt{2 R T \ln \frac{P}{P_0}} \tag{2.84}$$

It can be shown that the pressure ratio is

$$\frac{P_2}{P_1} = e^{\frac{k({M_1}^2 - {M_2}^2)}{2}} = \left(\frac{e^{{M_1}^2}}{e^{{M_2}^2}} \right)^{\frac{k}{2}} \tag{2.85}$$

As opposed to the adiabatic case ($T_0 = constant$) in the isothermal flow the stagnation temperature ratio can be expressed

$$\frac{T_{01}}{T_{02}} = \frac{\overset{1}{\cancel{T_1}}}{\cancel{T_2}} \frac{\left(1 + \frac{k-1}{2}{M_1}^2\right)}{\left(1 + \frac{k-1}{2}{M_2}^2\right)} = \frac{\left(1 + \frac{k-1}{2}{M_1}^2\right)}{\left(1 + \frac{k-1}{2}{M_2}^2\right)} \tag{2.86}$$

Combining equation mass conservation with equation (2.85) yields

$$\frac{A_2}{A_1} = \frac{M_1}{M_2} \left(\frac{\mathsf{e}^{{M_2}^2}}{\mathsf{e}^{{M_1}^2}} \right)^{\frac{k}{2}} \tag{2.87}$$

[2] The one dimensional momentum equation for steady state is $U dU/dx = -dP/dx + 0$(other effects) which are neglected here.

The change in the stagnation pressure can be expressed as

$$\frac{P_{02}}{P_{01}} = \frac{P_2}{P_1}\left(\frac{1+\frac{k-1}{2}{M_2}^2}{1+\frac{k-1}{2}{M_1}^2}\right)^{\frac{k}{k-1}} = \left[\frac{\mathsf{e}^{{M_1}^2}}{\mathsf{e}^{{M_1}^2}}\right]^{\frac{k}{2}} \tag{2.88}$$

The critical point, at this stage, is unknown (at what Mach number the nozzle is choked is unknown) so there are two possibilities: the choking point or $M = 1$ to normalize the equation. Here the critical point defined as the point where$M = 1$ so results can be compared to the adiabatic case and denoted by star. Again it has to be emphasized that this critical point is not really related to physical critical point but it is only an arbitrary definition. The true critical point is when flow is choked and the relationship between two will be presented.

The critical pressure ratio can be obtained from (2.85) to read

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \mathsf{e}^{\frac{(1-M^2)k}{2}} \tag{2.89}$$

Equation (2.87) is reduced to obtained the critical area ratio writes

$$\frac{A}{A^*} = \frac{1}{M}\mathsf{e}^{\frac{(1-M^2)k}{2}} \tag{2.90}$$

Similarly the stagnation temperature reads

$$\frac{T_0}{{T_0}^*} = \frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)^{\frac{k}{k-1}}}{k+1} \tag{2.91}$$

Finally, the critical stagnation pressure reads

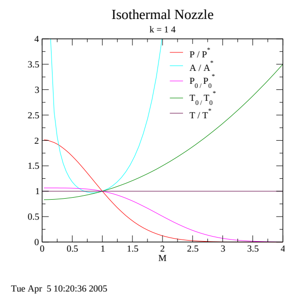

Fig. -2.7. Various ratios as a function of Mach number for isothermal Nozzle

$$\frac{P_0}{{P_0}^*} = \mathsf{e}^{\frac{(1-M^2)k}{2}}\left(\frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)}{k+1}\right)^{\frac{k}{k-1}} \tag{2.92}$$

The maximum value of stagnation pressure ratio is obtained when $M = 0$ at which is

$$\left.\frac{P_0}{{P_0}^*}\right|_{M=0} = \mathsf{e}^{\frac{k}{2}}\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{2.93}$$

For specific heats ratio of $k = 1.4$, this maximum value is about two. It can be noted that the stagnation pressure is monotonically reduced during this process.

Of course in isothermal process $T = T^*$. All these equations are plotted in Figure (2.7). From the Figure 2.7 it can be observed that minimum of the curve A/A^* isn't on $M = 1$. The minimum of the curve is when area is minimum and at the point where the flow is choked. It should be noted that the stagnation temperature is not constant as in the adiabatic case and the critical point is the only one constant.

The mathematical procedure to find the minimum is simply taking the derivative and equating to zero as the following

$$\frac{d\left(\frac{A}{A^*}\right)}{dM} = \frac{kM^2 e^{\frac{k(M^2-1)}{2}} - e^{\frac{k(M^2-1)}{2}}}{M^2} = 0 \tag{2.94}$$

Equation (2.94) simplified to

$$kM^2 - 1 = 0 \leadsto M = \frac{1}{\sqrt{k}} \tag{2.95}$$