Note:

CHAPTER 11: APPENDIX

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This chapter is part of the textbook:

"Basics of Fluid Mechanics"

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APPENDIX A

The Mathematics Backgrounds for Fluid Mechanics

In this appendix a review of selected topics in mathematics related to fluid mechanics is presented. These topics are present so that one with some minimal background could deal with the mathematics that encompass within basic fluid mechanics. Hence without additional reading, this book on fluid mechanics issues could be read by most readers. This appendix condenses material that spread in many various textbooks some of which are advance. Furthermore, some of the material appears in specialty books such as third order differential equations (and thus it is expected that the student is not familiar with this material.). There is very minimal original material which appears without proofs. The material is not presented in "educational" order but in importance order.

A.1 Vectors

Vector is a quantity with direction as oppose to scalar. The length of the vector in Cartesian coordinates (the coordinates system is relevant) is

$$\|\boldsymbol{U}\| = \sqrt{{U_x}^2 + {U_y}^2 + {U_z}^2} \qquad \text{(A.1)}$$

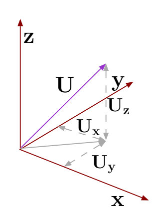

Fig. -A.1. Vector in Cartesian coordinates system.

Vector can be normalized and in Cartesian coordinates depicted in Figure A.1 where U_x is the vector component in the x direction, U_y is the vector component in the y direction, and U_z is the vector component in the z direction. Thus, the unit vector is

$$\widehat{\boldsymbol{U}} = \frac{\boldsymbol{U}}{\|\mathbf{U}\|} = \frac{U_x}{\|\boldsymbol{U}\|}\hat{\boldsymbol{i}} + \frac{U_y}{\|\boldsymbol{U}\|}\hat{\boldsymbol{j}} + \frac{U_z}{\|\boldsymbol{U}\|}\hat{\boldsymbol{k}} \tag{A.2}$$

and general orthogonal coordinates

$$\widehat{\boldsymbol{U}} = \frac{\boldsymbol{U}}{\|\mathbf{U}\|} = \frac{U_1}{\|\boldsymbol{U}\|}\mathbf{h_1} + \frac{U_2}{\|\boldsymbol{U}\|}\mathbf{h_2} + \frac{U_3}{\|\boldsymbol{U}\|}\mathbf{h_3} \tag{A.3}$$

Vectors have some what similar rules to scalars which will be discussed in the next section.

A.1.1 Vector Algebra

Vectors obey several standard mathematical operations which are applicable to scalars. The following are vectors, $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ and for in this discussion a and b are scalars. Then the following can be said

1. $(\boldsymbol{U}+\boldsymbol{V})+\boldsymbol{W} = (\boldsymbol{U}+\boldsymbol{V}+\boldsymbol{W}) = \boldsymbol{U}+(\boldsymbol{V}+\boldsymbol{W})$
2. $\boldsymbol{U}+\boldsymbol{V} = \boldsymbol{V}+\boldsymbol{U}$
3. Zero vector is such that $\boldsymbol{U}+\mathbf{0} = \boldsymbol{U}$
4. Additive inverse $\boldsymbol{U}-\boldsymbol{U} = 0$
5. $a\,(\boldsymbol{U}+\boldsymbol{V}) = a\,\boldsymbol{U} + a\,\boldsymbol{V}$
6. $a\,(b\,\boldsymbol{U}) = a\,b\,\boldsymbol{U}$

The multiplications and the divisions have somewhat different meaning in a scalar operations. There are two kinds of multiplications for vectors. The first multiplication is the "dot" product which is defined by equation (A.4). The results of this multiplication is scalar but has no negative value as in regular scalar multiplication.

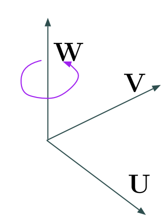

Fig. -A.2. The right hand rule, multiplication of $\boldsymbol{U} \times \boldsymbol{V}$ results in $\boldsymbol{W}$.

$$\boldsymbol{U}\cdot\boldsymbol{V} = \overbrace{|\boldsymbol{U}|\cdot|\boldsymbol{V}|}^{\text{regular scalar multiplication}} \; \cos\overbrace{(\angle(\boldsymbol{U},\boldsymbol{V}))}^{\text{angle between vectors}} \tag{A.4}$$

The second multiplication is the "cross" product which in vector as opposed to a scalar as in the "dot" product. The "cross" product is defined in an orthogonal coordinate ($\widehat{h_1}$, $\widehat{h_2}$, and $\widehat{h_3}$) as

$$\boldsymbol{U}\times\boldsymbol{V} = |\boldsymbol{U}|\cdot|\boldsymbol{V}|\sin\overbrace{(\angle(\boldsymbol{U},\boldsymbol{V}))}^{angle}\,\widehat{\boldsymbol{n}} \tag{A.5}$$

where θ is the angle between $\boldsymbol{U}$ and $\boldsymbol{V}$, and $\widehat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{U}$ and $\boldsymbol{V}$ which obeys the right hand rule. The right hand rule is referred to the direction of resulting vector. Note that $\boldsymbol{U}$ and $\boldsymbol{V}$ are not necessarily orthogonal. Additionally note that order of multiplication is significant. This multiplication has a negative value which means that it is a change of the direction.

One of the consequence of this definitions in Cartesian coordinates is

$$\widehat{\boldsymbol{i}}^2 = \widehat{\boldsymbol{j}}^2 = \widehat{\boldsymbol{k}}^2 = 0 \tag{A.6}$$

In general for orthogonal coordinates this condition is written as

$$\widehat{\boldsymbol{h_1}} \times \widehat{\boldsymbol{h_1}} = \widehat{\boldsymbol{h_1}}^2 = \widehat{\boldsymbol{h_2}}^2 = \widehat{\boldsymbol{h_3}}^2 = 0 \tag{A.7}$$

where $\boldsymbol{h_i}$ is the unit vector in the orthogonal system.

In right hand orthogonal coordinate system

$$\begin{array}{ll} \widehat{\boldsymbol{h_1}} \times \widehat{\boldsymbol{h_2}} = \widehat{\boldsymbol{h_3}} & \widehat{\boldsymbol{h_2}} \times \widehat{\boldsymbol{h_1}} = -\widehat{\boldsymbol{h_3}} \\ \widehat{\boldsymbol{h_2}} \times \widehat{\boldsymbol{h_3}} = \widehat{\boldsymbol{h_1}} & \widehat{\boldsymbol{h_3}} \times \widehat{\boldsymbol{h_2}} = -\widehat{\boldsymbol{h_1}} \\ \widehat{\boldsymbol{h_3}} \times \widehat{\boldsymbol{h_1}} = \widehat{\boldsymbol{h_2}} & \widehat{\boldsymbol{h_1}} \times \widehat{\boldsymbol{h_3}} = -\widehat{\boldsymbol{h_2}} \end{array} \tag{A.8}$$

The “cross” product can be written as

$$\boldsymbol{U} \times \boldsymbol{V} = (U_2\,V_3 - U_3\,V_2)\,\widehat{\boldsymbol{h_1}} + (U_3\,V_1 - U_1\,V_3)\,\widehat{\boldsymbol{h_2}} + (U_1\,V_2 - U_2\,V_1)\,\widehat{\boldsymbol{h_3}} \tag{A.9}$$

Equation (A.9) in matrix form as

$$\boldsymbol{U} \times \boldsymbol{V} = \begin{pmatrix} \widehat{\boldsymbol{h_1}} & \widehat{\boldsymbol{h_2}} & \widehat{\boldsymbol{h_3}} \\ U_2 & U_2 & U_3 \\ V_2 & V_2 & V_3 \end{pmatrix} \tag{A.10}$$

The most complex of all these algebraic operations is the division. The multiplication in vector world have two definition one which results in a scalar and one which results in a vector. Multiplication combinations shows that there are at least four possibilities of combining the angle with scalar and vector. The reason that these current combinations, that is scalar associated with $\cos\theta$ vectors is associated with $\sin\theta$, is that these combinations have physical meaning. The previous experience is that help to define multiplication help to definition the division. The number of the possible combinations of the division is very large. For example, the result of the division can be a scalar combined or associated with the angle (with $\cos$ or $\sin$), or vector with the angle, etc. However, these above four combinations are not the only possibilities (not including the left hand system). It turn out that these combinations have very little[1]

[1] This author did find any physical meaning these combinations but there could be and those the word “little” is used.

physical meaning. Additional possibility is that every combination of one vector element is divided by the other vector element. Since every vector element has three possible elements the total combination is $9 = 3 \times 3$. There at least are two possibilities how to treat these elements. It turned out that combination of three vectors has a physical meaning. The three vectors have a need for additional notation such of vector of vector which is referred to as a tensor. The following combination is commonly suggested

$$\frac{\boldsymbol{U}}{\boldsymbol{V}} = \begin{pmatrix} \frac{U_1}{V_1} & \frac{U_2}{V_1} & \frac{U_3}{V_1} \\ \frac{U_1}{V_2} & \frac{U_2}{V_2} & \frac{U_3}{V_2} \\ \frac{U_1}{V_3} & \frac{U_2}{V_3} & \frac{U_3}{V_3} \end{pmatrix} \tag{A.11}$$

One such example of this division is the pressure which the explanation is commonality avoided or eliminated from the fluid mechanics books including the direct approach in this book.

This tenser or the matrix can undergo regular linear algebra operations such as finding the eigenvalue values and the eigen "vectors." Also note the multiplying matrices and inverse matrix are also available operation to these tensors.

A.1.2 Differential Operators of Vectors

Differential operations can act on scalar functions as well on vector and vector functions. More differential operations can on scalar function can results in vector or vector function. In multivariate calculus, derivatives of different directions can represented as a vector or vector function. A compact presentation is a common way to handle the mathematics which simplify the calculations and explanations. One of these operations is nabla operator sometimes also called the "del operator." This operator is a differential vector. For example, in Cartesian coordinates the operation is

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{A.12}$$

Where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are denoting unit vectors in the $x, y,$ and z directions, respectively. Many of the operations of vector world, such as, the gradient, divergence, the curl, and the Laplacian are based or could be constructed from this single operator.

Gradient

This operation acts on a scalar function and results in a vector whose components are derivatives in the principle directions of a coordinate system. A scalar function is a function that provide a valued based on the coordinates (in Cartesian coordinates x,y,z). For example, the temperature of the domain might be expressed as a scalar field.

$$\nabla = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z} \tag{A.13}$$

Divergence

The same idea that was discussed in vector section there are two kinds of multiplication in the vector world and two will be for the differential operators. The divergence is the similar to "dot" product which results in scalar. A vector domain (function) assigns a vector to each point such as velocity for example, N, for Cartesian coordinates is

$$N(x,y,z) = N_x(x,y,z)\hat{\mathbf{i}} + N_y(x,y,z)\hat{\mathbf{j}} + N_z(x,y,z)\hat{\mathbf{k}} \tag{A.14}$$

The *dot* product of these two vectors, in Cartesian coordinate is results in

$$\mathrm{div}\ \mathbf{N} = \nabla \cdot \mathbf{N} = \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z} \tag{A.15}$$

The divergence results in a scalar function which similar to the concept of the vectors multiplication of the vectors magnitude by the cosine of the angle between the vectors.

Curl

Similar to the "cross product" a similar operation can be defined for the nabla (note the "right hand rule" notation) for Cartesian coordinate as

$$\mathrm{curl}\,\boldsymbol{N} = \nabla \times \boldsymbol{N} = \begin{array}{l} \left(\dfrac{\partial N_z}{\partial y} - \dfrac{\partial N_y}{\partial z}\right)\hat{i} + \\ \left(\dfrac{\partial N_x}{\partial z} - \dfrac{\partial N_z}{\partial x}\right)\hat{j} + \left(\dfrac{\partial N_y}{\partial x} - \dfrac{\partial N_x}{\partial y}\right)\hat{k} \end{array} \tag{A.16}$$

Note that the result is a vector.

Laplacian

The new operation can be constructed from "dot" multiplication of the nabla. A gradient acting on a scalar field creates a vector field. Applying a divergence on the result creates a scalar field again. This combined operations is known as the "div grad" which is given in Cartesian coordinates by

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{A.17}$$

This combination is commonality denoted as ∇^2. This operator also referred as the Laplacian operator, in honor of Pierre-Simon Laplace (23 March 1749 – 5 March 1827).

d'Alembertian

As a super–set for four coordinates (very minimal used in fluid mechanics) and it reffed to as d'Alembertian or the wave operator, and it defined as

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial^2 t} \tag{A.18}$$

Divergence Theorem

Mathematicians call to or refer to a subset of The Reynolds Transport Theorem as the Divergence Theorem, or called it Gauss' Theorem (Carl Friedrich Gauss 30 April 1777 23 February 1855), In Gauss notation it is written as

$$\iiint_V (\nabla \cdot \boldsymbol{N})\, dV = \oiint_A \boldsymbol{N} \cdot \boldsymbol{n}\, dA \tag{A.19}$$

In Gauss-Ostrogradsky Theorem (Mikhail Vasilievich Ostrogradsky (September 24, 1801 – January 1, 1862). The notation is a bit different from Gauss and it is written in Ostrogradsky notation as

$$\int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\, dy\, dz = \int\int_\Sigma (Pp + Qq + Rr)\, d\Sigma \tag{A.20}$$

Note the strange notation of "Σ" which refers to the area. This theorem is applicable for a fix control volume and the derivative can enters into the integral. Many engineering class present this theorem as a theorem on its merit without realizing that it is a subset of Reynolds Transport Theorem. This subset can further produces several interesting identities. If $\boldsymbol{N}$ is a gradient of a scalar field $\Pi(x, y, z)$ then it can insert into identity to produce

$$\iiint_V (\nabla \cdot (\nabla \boldsymbol{\Phi}))\, dV = \iiint_V \left(\nabla^2 \Phi\right) dV = \oiint_A \nabla \boldsymbol{\Phi} \cdot \boldsymbol{n}\, dA \tag{A.21}$$

Since the definition of $\nabla \boldsymbol{\Phi} = \boldsymbol{N}$.

Special case of equation (A.21) for harmonic function (solutions Laplace equation see[2] Harminic functions) then the left side vanishes which is useful identity for ideal flow analysis. This results reduces equation, normally for steady state, to a balance of the fluxes through the surface. Thus, the harmonic functions can be added or subtracted because inside the volume these functions contributions is eliminated throughout the volume.

A.1.3 Differentiation of the Vector Operations

The vector operation sometime fell under (time or other) derivative. The basic of these relationships is explored. A vector is made of the several scalar functions such as

$$\vec{\boldsymbol{R}} = f_1(x_1, x_2, x_3, \cdots)\widehat{\boldsymbol{e}_1} + f_2(x_1, x_2, x_3, \cdots)\widehat{\boldsymbol{e}_2} + f_3(x_1, x_2, x_3, \cdots)\widehat{\boldsymbol{e}_3} + \cdots \tag{A.22}$$

where $\widehat{\boldsymbol{e}_i}$ is the unit vector in the i direction. The cross and dot products when the come under differentiation can be look as scalar. For example, the dot product of operation

[2] for more information
`http://math.fullerton.edu/mathews/c2003/HarmonicFunctionMod.html`

$\boldsymbol{R}\cdot\boldsymbol{S} = (x\hat{i} + y^2\hat{j})\cdot(\sin x\hat{i} + exp(y)\hat{j})$ can be written as

$$\frac{d\,(\boldsymbol{R}\cdot\boldsymbol{S})}{dt} = \frac{d}{dt}\left(\left(x\hat{i} + y^2\hat{j}\right)\cdot\left(\sin x\hat{i} + exp(y)\hat{j}\right)\right)$$

It can be noticed that

$$\frac{d\,(\boldsymbol{R}\cdot\boldsymbol{S})}{dt} = \frac{d\left(x\,\sin x + y^2\,exp(y)\right)}{dt} = \frac{d\,x}{dt}\sin x + \frac{d\,\sin x}{dt} + \frac{d\,y^2}{dt}\exp(y) + \frac{d\,y^2}{dt}exp(y)$$

It can be noticed that the manipulation of the simple above example obeys the regular chain role. Similarly, it can done for the cross product. The results of operations of two vectors is similar to regular multiplication since the vectors operation obey "regular" addition and multiplication roles, the chain role is applicable. Hence the chain role apply for dot operation,

$$\frac{d}{dt}\left(\boldsymbol{R}\cdot\boldsymbol{S}\right) = \frac{d\boldsymbol{R}}{dt}\cdot\boldsymbol{S} + \frac{d\boldsymbol{S}}{dt}\cdot\boldsymbol{R} \tag{A.23}$$

And the the chain role for the cross operation is

$$\frac{d}{dt}\left(\boldsymbol{R}\times\boldsymbol{S}\right) = \frac{d\boldsymbol{R}}{dt}\times\boldsymbol{S} + \frac{d\boldsymbol{S}}{dt}\times\boldsymbol{R} \tag{A.24}$$

It follows that derivative (notice the similarity to scalar operations) of

$$\frac{d}{dt}\left(\boldsymbol{R}\cdot\boldsymbol{R}\right) = 2\,\boldsymbol{R}\frac{d\,\boldsymbol{R}}{at}$$

There are several identities that related to location, velocity, and acceleration. As in operation on scalar time derivative of dot or cross of constant velocity is zero. Yet, the most interesting is

$$\frac{d}{dt}\left(\boldsymbol{R}\times\boldsymbol{U}\right) = \boldsymbol{U}\times\boldsymbol{U} + \boldsymbol{R}\times\frac{d\boldsymbol{U}}{dt} \tag{A.25}$$

The first part is zero because the cross product with itself is zero. The second part is zero because Newton law (acceleration is along the path of R).

A.1.3.1 Orthogonal Coordinates

These vectors operations can appear in different orthogonal coordinates system. There are several orthogonal coordinates which appears in fluid mechanics operation which include this list: Cartesian coordinates, Cylindrical coordinates, Spherical coordinates, Parabolic coordinates, Parabolic cylindrical coordinates Paraboloidal coordinates, Oblate spheroidal coordinates, Prolate spheroidal coordinates, Ellipsoidal coordinates, Elliptic cylindrical coordinates, Toroidal coordinates, Bispherical coordinates, Bipolar cylindrical coordinates Conical coordinates, Flat-ring cyclide coordinates, Flat-disk cyclide coordinates, Bi-cyclide coordinates and Cap-cyclide coordinates. Because there are so many coordinates system is reasonable to develop these operations for any for any coordinates system. Three common systems typical to fluid mechanics will be presented and followed by a table and methods to present all the above equations.

Cylindrical Coordinates

The cylindrical coordinates are commonality used in situations where there is line of symmetry or kind of symmetry. This kind situations occur in pipe flow even if the pipe is not exactly symmetrical. These coordinates reduced the work, in most cases, because problem is reduced a two dimensions. Historically, these coordinate were introduced for geometrical problems about 2000 years ago[3]. The cylindrical coordinates are shown in Figure A.3. In the figure shows that the coordinates are r, θ, and z. Note that unite coordinates are denoted as $\widehat{r}$, $\widehat{\theta}$, and $\widehat{z}$. The meaning of $\overrightarrow{r}$ and $\widehat{r}$ are different. The first one represents the vector that is the direction of $\widehat{r}$ while the second is the unit vector in the direction of the coordinate r. These three different rs are some what similar to any of the Cartesian coordinate. The second coordinate θ has unite coordinate $\widehat{\theta}$. The new concept here is the length factor. The coordinate θ is angle. In this book the dimensional chapter shows that in physics that derivatives have to have same units in order to compare them or use them. Conversation of the angel to units of length is done by length factor which is, in this case, r. The conversion between the Cartesian coordinate and the Cylindrical is

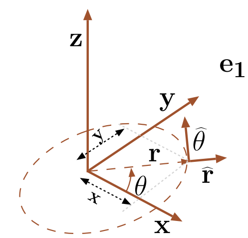

Fig. -A.3. Cylindrical Coordinate System.

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x} \qquad z = z \tag{A.26}$$

The reverse transformation is

$$x = r\,\cos\theta \qquad y = r\,\sin\theta \qquad z = z \tag{A.27}$$

The line element and volume element are

$$ds = \sqrt{dr^2 + \left(r\,d\theta\right)^2 + dz^2} \qquad dr\,r\,d\theta\,dz \tag{A.28}$$

The gradient in cylindrical coordinates is given by

$$\nabla = \widehat{r}\frac{\partial}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \widehat{z}\frac{\partial}{\partial z} \tag{A.29}$$

[3]Coolidge, Julian (1952). "The Origin of Polar Coordinates". American Mathematical Monthly 59: 7885. `http://www-history.mcs.st-and.ac.uk/Extras/Coolidge_Polars.html`. Note the advantage of cylindrical (polar) coordinates in description of geometry or location relative to a center point.

The curl is written

$$\nabla \times \boldsymbol{N} = \left(\frac{1}{r}\frac{\partial N_z}{\partial \theta} - \frac{\partial N_\theta}{\partial z} \right) \widehat{\boldsymbol{r}} + \left(\frac{\partial N_r}{\partial z} - \frac{\partial N_z}{\partial r} \right) \widehat{\boldsymbol{\theta}} + \tag{A.30}$$

$$\frac{1}{r}\left(\frac{\partial \left(r\, N_\theta \right)}{\partial r} - \frac{\partial N_\theta}{\partial \theta} \right) \widehat{\boldsymbol{z}} \tag{A.31}$$

The Laplacian is defined by

$$\nabla \cdot \nabla = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{A.32}$$

Spherical Coordinates

The spherical coordinates system is a three-dimensional coordinates which is improvement or further modifications of the cylindrical coordinates. Spherical system used for cases where spherical symmetry exist. In fluid mechanics such situations exist in bubble dynamics, boom explosion, sound wave propagation etc. A location is represented by a radius and two angles. Note that the first angle (azimuth or longitude) θ range is between $0 < \theta < 2\pi$ while the second angle (colatitude) is only $0 < \phi < \pi$. The radius is the distance between the origin and the location. The first angle between projection on $x - y$ plane and the positive x–axis. The second angle is between the positive y–axis and the vector as shown in Figure A.4.

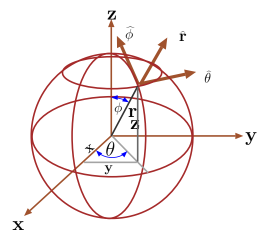

Fig. -A.4. Spherical Coordinate System.

The conversion between Cartesian coordinates to Spherical coordinates

$$x = r\, \sin\phi\, \cos\theta \qquad y = r\, \sin\phi\, \sin\theta \qquad z = r\, \cos\phi \tag{A.33}$$

The reversed transformation is

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \phi = \arccos\left(\frac{z}{r} \right) \tag{A.34}$$

Line element and element volume are

$$ds = \sqrt{dr^2 + \left(r\, \cos\theta\, d\theta\right)^2 + \left(r\, \sin\theta\, d\phi\right)^2} \qquad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{A.35}$$

The gradient is

$$\nabla = \widehat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \widehat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{A.36}$$

The divergence in spherical coordinate is

$$\nabla \cdot \boldsymbol{N} = \frac{1}{r^2}\frac{\partial \left(r^2 N_r \right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \left(N_\theta \sin\theta \right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial N_\phi}{\partial \phi} \tag{A.37}$$

The curl in spherical coordinates is

$$\nabla \times \boldsymbol{N} = \frac{1}{r\sin\theta}\left(\frac{\partial\left(N_\phi \sin\theta\right)}{\partial\theta} - \frac{\partial N_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial N_r}{\partial\phi} - \frac{\partial\left(r N_\phi\right)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial\left(r N_\theta\right)}{\partial r} - \frac{\partial N_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \tag{A.38}$$

The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{A.39}$$

General Orthogonal Coordinates

There are several orthogonal system and general form is needed. The notation for the presentation is required general notation of the units vectors is $\widehat{e_i}$ and coordinates distance coefficient is h_i where i is 1,2,3. The coordinates distance coefficient is the change the differential to the actual distance. For example in cylindrical coordinates, the unit vectors are: $\hat{r}$, $\hat{\theta}$, and $\hat{z}$. The units $\hat{r}$ and $\hat{z}$ are units with length. However, $\hat{\theta}$ is lengthens unit vector and the coordinate distance coefficient in this case is r. As in almost all cases, there is dispute what the proper notation for these coefficients. In mathematics it is denoted as q while in engineering is denotes h. Since it is engineering book the h is adapted. Also note that the derivative of the coordinate in the case of cylindrical coordinate is $\partial\theta$ and unit vector is $\hat{\theta}$. While the θ is the same the meaning is different and different notations need. The derivative quantity will be denoted by q superscript.

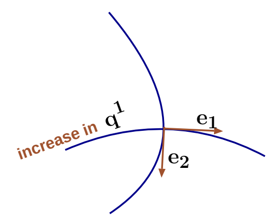

Fig. -A.5. The general Orthogonal with unit vectors.

The length of

$$d\ell^2 = \sum_{i=1}^{d}\left(h_k dq^k\right)^2 \tag{A.40}$$

The nabla operator in general orthogonal coordinates is

$$\nabla = \frac{\widehat{e_1}}{h_1}\frac{\partial}{\partial q^1} + \frac{\widehat{e_2}}{h_2}\frac{\partial}{\partial q^2} + \frac{\widehat{e_3}}{h_3}\frac{\partial}{\partial q^3} \tag{A.41}$$

Gradient

The gradient in general coordinate for a scalar function $\boldsymbol{T}$ is the nabla operator in general orthogonal coordinates as

$$\nabla\boldsymbol{T} = \frac{\widehat{e_1}}{h_1}\frac{\partial\boldsymbol{T}}{\partial q^1} + \frac{\widehat{e_2}}{h_2}\frac{\partial\boldsymbol{T}}{\partial q^2} + \frac{\widehat{e_3}}{h_3}\frac{\partial\boldsymbol{T}}{\partial q^3} \tag{A.42}$$

The divergence of a vector equals

$$\nabla \cdot \boldsymbol{N} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q^1}\left(N_1 h_2 h_3\right) + \frac{\partial}{\partial q^2}\left(N_2 h_3 h_1\right) + \frac{\partial}{\partial q^3}\left(N_3 h_1 h_2\right)\right]. \tag{A.43}$$

For general orthogonal coordinate system the curl is

$$\begin{gathered}\nabla \times \boldsymbol{N} = \frac{\widehat{e_1}}{h_2\, h_3}\left[\frac{\partial}{\partial q^2}\left(h_3\, N_3\right) - \frac{\partial}{\partial q^3}\left(h_2\, N_2\right)\right] + \\ \frac{\widehat{e_2}}{h_3\, h_1}\left[\frac{\partial}{\partial q^3}\left(h_1\, N_1\right) - \frac{\partial}{\partial q^1}\left(h_3\, N_3\right)\right] + \frac{\widehat{e_3}}{h_1\, h_2}\left[\frac{\partial}{\partial q^1}\left(h_2 N_2\right) - \frac{\partial}{\partial q^2}\left(h_1 N_1\right)\right]\end{gathered} \tag{A.44}$$

The Laplacian of a scalar equals

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q^1}\right) + \frac{\partial}{\partial q^2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial q^2}\right) + \frac{\partial}{\partial q^3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial q^3}\right)\right] \tag{A.45}$$

The following table showing the different values for selected orthogonal system.

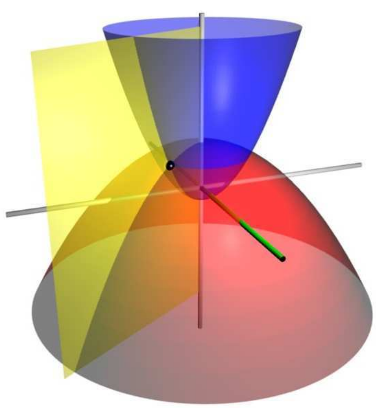

Fig. -A.6. Parabolic coordinates by user WillowW using Blender.

Table -A.1. Orthogonal coordinates systems (under construction please ignore)

Orthogonal coordinates systems	Remarks	h			q		
name		1	2	3	1	2	3
Cartesian	standard	1	1	1	x	y	z
Cylindrical	common	1	r	1	r	θ	z
Spherical	common	1	r	$r\cos\theta$	r	θ	φ
Paraboloidal	?	$\sqrt{u^2+v^2}$	$\sqrt{u^2+v^2}$	$u\,v$	u	v	θ
Ellipsoidal	?				λ	μ	ν

A.2 Ordinary Differential Equations (ODE)

In this section a brief summary of ODE is presented. It is not intent to be a replacement to a standard textbook but as a quick reference. It is suggested that the reader interested in depth information should read "Differential Equations and Boundary Value Problems" by Boyce de–Prima or any other book in this area. Ordinary differential equations are defined by the order of the highest derivative. If the highest derivative is first order the equation is referred as first order differential equation etc. Note that the derivatives are integers e.g. first derivative, second derivative etc[4]. ODE are categorized into linear and non-linear equations. The meaning of linear equation is that the operation is such that

$$a\,L\left(u_1\right) + b\,L\left(u_2\right) = L\left(a\,u_1 + b\,u_2\right) \tag{A.46}$$

An example of such linear operation $L = \frac{d}{dt} + 1$ acting on y is $\frac{dy_1}{dt} + y_1$. Or this operation on y_2 is $\frac{dy_2}{dt} + y_2$ and the summation of operation the sum operation of $L(y_1 + y_2) = \frac{y_1+y_2}{dt} + y_1 + y_2$.

A.2.1 First Order Differential Equations

As expect, the first ODEs are easier to solve and they are the base for equations of higher order equation. The first order equations have several forms and there is no one solution fit all but families of solutions. The most general form is

$$f\left(u, \frac{du}{dt}, t\right) = 0 \tag{A.47}$$

[4]Note that mathematically, it is possible to define fraction of derivative. However, there is no physical meaning to such a product according to this author believe.

Sometimes equation (A.47) can be simplified to the first form as

$$\frac{du}{dt} = F(t, u) \tag{A.48}$$

A.2.2 Variables Separation or Segregation

In some cases equation (A.48) can be written as $F(t, u) = X(t)\,U(u)$. In that case it is said that F is spreadable and then equation (A.48) can be written as

$$\frac{du}{U(u)} = X(t)dt \tag{A.49}$$

Equation can be integrated either analytically or numerically and the solution is

$$\int \frac{du}{U(u)} = \int X(t)dt \tag{A.50}$$

The limits of the integral is (are) the initial condition(s). The initial condition is the value the function has at some points. The name initial condition is used because the values are given commonly at initial time.

Example A.1:
Solve the following equation

$$\frac{du}{dt} = u\,t \tag{1.I.a}$$

with the initial condition $u(t = 0) = u_0$.

SOLUTION

The solution can be obtained by the variable separation method. The separation yields

$$\frac{du}{u} = t\,dt \tag{1.I.b}$$

The integration of equation (1.I.b) becomes

$$\int \frac{du}{u} = \int t\,dt \Longrightarrow \ln(u) + \ln(c) = \frac{t^2}{2} \tag{1.I.c}$$

Equation (1.I.c) can be transferred to

$$u = c\,e^{t^2} \tag{1.I.d}$$

For the initial condition of $u(0) = u_0$ then

$$u = u_0\,e^{t^2} \tag{1.I.e}$$

End Solution

A.2.2.1 The Integral Factor Equations

Another method is referred to as integration factor which deals with a limited but very important class of equations. This family is part of a linear equations. The general form of the equation is

$$\frac{dy}{dx} + g(x)\,y = m(x) \tag{A.51}$$

Multiplying equation (A.51) by unknown function $N(x)$ transformed it to

$$N(x)\,\frac{dy}{dx} + N(x)\,g(x)\,y = N(x)m(x) \tag{A.52}$$

What is needed from $N(x)$ is to provide a full differential such as

$$N(x)\,\frac{dy}{dx} + N(x)\,g(x)\,y = \frac{d\,[N(x)\,g(x)\,y]}{dx} \tag{A.53}$$

This condition (note that the previous methods is employed here) requires that

$$\frac{d\,N(x)}{dx} = N(x)\,g(x) \Longrightarrow \frac{d\,N(x)}{N(x)} = g(x)\,dx \tag{A.54}$$

Equation (A.54) is integrated to be

$$\ln\left(N(x)\right) = \int g(x)dx \Longrightarrow N(x) = \mathsf{e}^{\displaystyle\int g(x)dx} \tag{A.55}$$

Using the differentiation chain rule provides

$$\frac{d\,N(x)}{dx} = \overbrace{\mathsf{e}^{\displaystyle\int g(x)dx}}^{\frac{dv}{du}} \overbrace{g(x)}^{\frac{du}{dx}} \tag{A.56}$$

which indeed satisfy equation (A.53). Thus equation (A.52) becomes

$$\frac{d\,[N(x)\,g(x)\,y]}{dx} = N(x)\,m(x) \tag{A.57}$$

Multiplying equation (A.57) by dx and integrating results in

$$N(x)\,g(x)\,y = \int N(x)\,m(x)\,dx \tag{A.58}$$

The solution is then

$$y = \frac{\displaystyle\int N(x)\,m(x)\,dx}{g(x)\,\underbrace{\mathsf{e}^{\int g(x)dx}}_{N(x)}} \tag{A.59}$$

A special case of $g(t) = constant$ is shown next.

Example A.2:
Find the solution for a typical problem in fluid mechanics (the problem of Stoke flow or the parachute problem) of

$$\frac{dy}{dx} + y = 1$$

SOLUTION

Substituting $m(x) = 1$ and $g(x) = 1$ into equation (A.59) provides

$$y = \mathrm{e}^{-x}\, (\mathrm{e}^{x} + c) = 1 + c\,\mathrm{e}^{-x}$$

End Solution

A.2.3 Non–Linear Equations

Non-Linear equations are equations that the power of the function or the function derivative is not equal to one or their combination. Many non linear equations can be transformed into linear equations and then solved with the linear equation techniques. One such equation family is referred in the literature as the Bernoulli Equations[5]. This equation is

$$\frac{du}{dt} + m(t) u = n(t) \overbrace{u^p}^{\text{non–linear part}} \tag{A.60}$$

The transformation $v = u^{1-p}$ turns equation (A.60) into a linear equation which is

$$\frac{dv}{dt} + (1-p)\, m(t)\, v = (1-p)\, n(t) \tag{A.61}$$

The linearized equation can be solved using the linear methods. The actual solution is obtained by reversed equation which transferred solution to

$$u = v^{(p-1)} \tag{A.62}$$

Example A.3:
Solve the following Bernoulli equation

$$\frac{du}{dt} + t^2\, u = \sin(t)\, u^3 \tag{1.III.a}$$

[5]Not to be confused with the Bernoulli equation without the s that referred to the energy equation.

Solution

The transformation is

$$v = u^2 \tag{1.III.b}$$

Using the definition (1.III.b) equation (1.III.a) becomes

$$\frac{dv}{dt} \overbrace{-2}^{1-p} t^2\, v = \overbrace{-2}^{1-p} sin(t) \tag{1.III.c}$$

The homogeneous solution of equation (1.III.c) is

$$u(t) = c e^{\frac{-t^3}{3}} \tag{1.III.d}$$

And the general solution is

$$u = e^{-\frac{t^3}{3}} \left(\overbrace{\int e^{\frac{t^3}{3}} \sin(t)\, dt}^{\text{private solution}} + c \right) \tag{1.III.e}$$

End Solution

A.2.3.1 Homogeneous Equations

Homogeneous function is given as

$$\frac{du}{dt} = f(u,t) = f(a\,u, a\,t) \tag{A.63}$$

for any real positive a. For this case, the transformation of $u = v\,t$ transforms equation (A.63) into

$$t\,\frac{dv}{dt} + v = f(1,v) \tag{A.64}$$

In another words if the substitution $u = v\,t$ is inserted the function f become a function of only v it is homogeneous function. Example of such case $u\prime = \left(u^3 - t^3\right)/t^3$ becomes $u\prime = \left(v^3 + 1\right)$. The solution is then

$$\ln|t| = \int \frac{dv}{f(1,v) - v} + c \tag{A.65}$$

Example A.4:

Solve the equation

$$\frac{du}{dt} = \sin\left(\frac{u}{t}\right) + \left(\frac{u^4 - t^4}{t^4}\right) \tag{1.IV.a}$$

SOLUTION

Substituting $u = v\,T$ yields

$$\frac{du}{dt} = \sin(v) + v^4 - 1 \tag{1.IV.b}$$

or

$$t\,\frac{dv}{dt} + v = \sin(v) + v^4 - 1 \Longrightarrow t\,\frac{dv}{dt} = \sin(v) + v^4 - 1 - v \tag{1.IV.c}$$

Now equation (1.IV.c) can be solved by variable separation as

$$\frac{dv}{\sin(v) + v^4 - 1 - v} = t\,dt \tag{1.IV.d}$$

Integrating equation (1.IV.d) results in

$$\int \frac{dv}{\sin(v) + v^4 - 1 - v} = \frac{t^2}{2} + c \tag{1.IV.e}$$

The initial condition can be inserted via the boundary of the integral.

End Solution

A.2.3.2 Variables Separable Equations

In fluid mechanics and many other fields there are differential equations that referred to variables separable equations. In fact, this kind of class of equations appears all over this book. For this sort equations, it can be written that

$$\frac{du}{dt} = f(t)g(u) \tag{A.66}$$

The main point is that $f(t)$ and be segregated from $g(u)$. The solution of this kind of equation is

$$\int \frac{du}{g(u)} = \int f(t)\,dt \tag{A.67}$$

Example A.5:
Solve the following ODE

$$\frac{du}{dt} = -u^2\,t^2 \tag{1.V.a}$$

SOLUTION

Segregating the variables to be

$$\int \frac{du}{u^2} = \int t^2\,dt \tag{1.V.b}$$

Integrating equation (1.V.b) transformed into

$$-\frac{1}{u} = \frac{t^3}{3} + c_1 \tag{1.V.c}$$

Rearranging equation (1.V.c) becomes

$$u = \frac{-3}{t^3 + c} \tag{1.V.d}$$

End Solution

A.2.3.3 Other Equations

There are equations or methods that were not covered by the above methods. There are additional methods such numerical analysis, transformation (like Laplace transform), variable substitutions, and perturbation methods. Many of these methods will be eventually covered by this appendix.

A.2.4 Second Order Differential Equations

The general idea of solving second order ODE is by converting them into first order ODE. One such case is the second order ODE with constant coefficients.

The simplest equations are with constant coefficients such as

$$a\,\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + c\,u = 0 \tag{A.68}$$

In a way, the second order ODE is transferred to first order by substituting the one linear operator to two first linear operators. Practically, it is done by substituting e^{st} where s is characteristic constant and results in the quadratic equation

$$a\,s^2 + b\,s + s = 0 \tag{A.69}$$

If $b^2 > 4\,a\,c$ then there are two unique solutions for the quadratic equation and the general solution form is

$$u = c_1\,e^{s_1\,t} + c_2\,e^{s_2\,t} \tag{A.70}$$

For the case of $b^2 = 4\,a\,c$ the general solution is

$$u = c_1\,e^{s_1\,t} + c_2\,t\,e^{s_1\,t} \tag{A.71}$$

In the case of $b^2 > 4\,a\,c$, the solution of the quadratic equation is a complex number which means that the solution has exponential and trigonometric functions as

$$u = c_1 e^{\alpha\,t}\cos(\beta t) + c_2 e^{\alpha\,t}\sin(\beta t) \tag{A.72}$$

Where the real part is

$$\alpha = \frac{-b}{2\,a} \tag{A.73}$$

and the imaginary number is

$$\beta = \frac{\sqrt{4\,a\,c - b^2}}{2\,a} \tag{A.74}$$

Example A.6:
Solve the following ODE

$$\frac{d^2u}{dt^2} + 7\,\frac{du}{dt} + 10\,u = 0 \tag{1.VI.a}$$

Solution

The characteristic equation is

$$s^2 + 7\,s + 10 = 0 \tag{1.VI.b}$$

The solution of equation (1.VI.b) are -2, and -5. Thus, the solution is

$$u = k_1\,e^{-2\,t} + k_2\,e^{-5\,t} \tag{1.VI.c}$$

End Solution

A.2.4.1 Non–Homogeneous Second ODE

Homogeneous equation are equations that equal to zero. This fact can be used to solve non-homogeneous equation. Equations that not equal to zero in this form

$$a\,\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + c\,u = l(x) \tag{A.75}$$

The solution of the homogeneous equation is zero that is the operation $L(u_h) = 0$, where L is Linear operator. The additional solution of $L(u_p)$ is the total solution as

$$L\left(u_{total}\right) = \overbrace{L\left(u_h\right)}^{=0} + L\left(u_p\right) \Longrightarrow u_{total} = u_h + u_p \tag{A.76}$$

Where the solution u_h is the solution of the homogeneous solution and u_p is the solution of the particular function $l(x)$. If the function on the right hand side is polynomial than the solution is will

$$u_{total} = u_h + \sum_{i=1}^{n} u_{p_i} \tag{A.77}$$

The linearity of the operation creates the possibility of adding the solutions.

Example A.7:
Solve the non-homogeneous equation

$$\frac{d^2u}{dt^2} - 5\,\frac{du}{dt} + 6\,u = t + t^2$$

SOLUTION

The homogeneous solution is

$$u(t) = c_1\,e^{2\,t} + c_1\,e^{3\,t} \tag{1.VII.a}$$

the particular solution for t is

$$u(t) = \frac{6\,t+5}{36} \tag{1.VII.b}$$

and the particular solution of the t^2 is

$$u(t) = \frac{18\,t^2 + 30\,t + 19}{108} \tag{1.VII.c}$$

The total solution is

$$u(t) = c_1\,e^{2\,t} + c_1\,e^{3\,t} + \frac{9\,t^2 + 24\,t + 17}{54} \tag{1.VII.d}$$

End Solution

A.2.5 Non–Linear Second Order Equations

Some of the techniques that were discussed in the previous section (first order ODE) can be used for the second order ODE such as the variable separation.

A.2.5.1 Segregation of Derivatives

If the second order equation

$$f(u, \dot{u}, \ddot{u}) = 0$$

can be written or presented in the form

$$f_1(u)\dot{u} = f_2\left(\dot{u}\right)\ddot{u} \tag{A.78}$$

then the equation (A.78) is referred to as a separable equation (some called it segregated equations). The derivative of $\dot{u}$ can be treated as a new function v and $\dot{v} = \ddot{u}$. Hence, equation (A.78) can be integrated

$$\int_{u_0}^{u} f_1(u)\dot{u} = \int_{\dot{u}_0}^{\dot{u}} f_2\left(\dot{u}\right)\ddot{u} = \int_{v_0}^{v} f_2\left(u\right)\dot{v} \tag{A.79}$$

The integration results in a first order differential equation which should be dealt with the previous methods. It can be noticed that the function initial condition is used twice; first with initial integration and second with the second integration. Note that the derivative initial condition is used once. The physical reason is that the equation represents a strong effect of the function at a certain point such surface tension problems. This equation family is not well discussed in mathematical textbooks[6].

Example A.8:
Solve the equation

$$\sqrt{u}\,\frac{du}{dt} - \sin\left(\frac{du}{dt}\right)\frac{d^2u}{dt^2} = 0$$

With the initial condition of $u(0) = 0$ *and* $\frac{du}{dt}\,(t = 0) = 0$ *What happen to the extra "*dt*"?*

SOLUTION

Rearranging the ODE to be

$$\sqrt{u}\frac{du}{\cancel{dt}} = \sin\left(\frac{du}{dt}\right)\frac{d}{\cancel{dt}}\left(\frac{du}{dt}\right) \tag{1.VIII.a}$$

Thus the extra dt is disappeared and equation (1.VIII.a) becomes

$$\int \sqrt{u}\,du = \int \sin\left(\frac{du}{dt}\right) d\left(\frac{du}{dt}\right) \tag{1.VIII.b}$$

and transformation to v is

$$\int \sqrt{u}\,du = \int \sin(v)\,dv \tag{1.VIII.c}$$

After the integration equation (1.VIII.c) becomes

$$\frac{2}{3}\left(u^{\frac{3}{2}} - {u_0}^{\frac{3}{2}}\right) = \cos(v_0) - \cos(v) = \cos\left(\frac{du_0}{dt}\right) - \cos\left(\frac{du}{dt}\right) \tag{1.VIII.d}$$

Equation (1.VIII.d) can be rearranged as

$$\frac{du}{dt} = \arcsin\left(\frac{2}{3}\left({u_0}^{\frac{3}{2}} - u^{\frac{3}{2}}\right) + \cos(v_0)\right) \tag{A.80}$$

Using the first order separation method yields

$$\int_0^t dt = \int_{u_0}^u \frac{du}{\arcsin\left(\frac{2}{3}\left(\underbrace{{u_0}^{\frac{3}{2}}}_{=0} - u^{\frac{3}{2}}\right) + \underbrace{\cos(v_0)}_{=1}\right)} \tag{A.81}$$

[6]This author worked (better word toyed) in (with) this area during his master but to his shame he did not produce any papers on this issue. The papers are still his drawer and waiting to a spare time.

The solution (A.81) shows that initial condition of the function is used twice while the initial of the derivative is used only once.

End Solution

A.2.5.2 Full Derivative Case Equations

Another example of special case or families of second order differential equations which is results of the energy integral equation derivations as

$$u - a\,u \left(\frac{du}{d\,t}\right) \left(\frac{d^{2u}}{d\,t^2}\right) = 0 \tag{A.82}$$

where a is constant. One solution is $u = k_1$ and the second solution is obtained by solving

$$\frac{1}{a} = \left(\frac{du}{d\,t}\right) \left(\frac{d^{2u}}{d\,t^2}\right) \tag{A.83}$$

The transform of $v = \frac{du}{dt}$ results in

$$\frac{1}{a} = v\frac{dv}{dt} \Longrightarrow \frac{dt}{a} = v\,dv \tag{A.84}$$

which can be solved with the previous methods.

Bifurcation to two solutions leads

$$\frac{t}{a} + c = \frac{1}{2}\,v^2 \Longrightarrow \frac{du}{dt} = \pm\sqrt{\frac{2\,t}{a} + c_1} \tag{A.85}$$

which can be integrated as

$$u = \int \pm\sqrt{\frac{2\,t}{a} + c_1}\;dt = \pm\frac{a}{3}\left(\frac{2\,t}{a} + c_1\right)^{\frac{3}{2}} + c_2 \tag{A.86}$$

A.2.5.3 Energy Equation ODE

It is non–linear because the second derivative is square and the function multiply the second derivative.

$$u\left(\frac{d^2u}{d\,t^2}\right) + \left(\frac{du}{d\,t}\right)^2 = 0 \tag{A.87}$$

It can be noticed that that c_2 is actually two different constants because the plus minus signs.

$$\frac{d}{dt}\left(u\,\frac{du}{d\,t}\right) = 0 \tag{A.88}$$

after integration

$$u\,\frac{du}{d\,t} = k_1 \tag{A.89}$$

Further rearrangement and integration leads to the solution which is

$$\frac{u^2}{2\,k_1} = t + k_2 \tag{A.90}$$

For non–homogeneous equation they can be integrated as well.

Example A.9:
Show that the solution of

$$u\left(\frac{d^2u}{d\,t^2}\right) + \left(\frac{du}{d\,t}\right)^2 + u = 0 \tag{1.IX.a}$$

is

$$-\frac{\sqrt{3}\displaystyle\int\frac{u}{\sqrt{3\,k_1 - u^3}}du}{\sqrt{2}} = t + k_2 \tag{1.IX.b}$$

$$\frac{\sqrt{3}\displaystyle\int\frac{u}{\sqrt{3\,k_1 - u^3}}du}{\sqrt{2}} = t + k_2 \tag{1.IX.c}$$

A.2.6 Third Order Differential Equation

There are situations where fluid mechanics[7] leads to third order differential equation. This kind of differential equation has been studied in the last 30 years to some degree. The solution to constant coefficients is relatively simple and will be presented here. Solution to more complicate linear equations with non constant coefficient (function of t) can be solved sometimes by Laplace transform or reduction of the equation to second order Olivier Vallee[8].

The general form for constant coefficient is

$$\frac{d^3u}{dt^3} + a\,\frac{d^2u}{dt^2} + b\,\frac{du}{dt} + c\,u = 0 \tag{A.91}$$

The solution is assumed to be of the form of e^{st} which general third order polonium. Thus, the general solution is depend on the solution of third order polonium. Third

[7]The unsteady energy equation in accelerated coordinate leads to a third order differential equation.
[8]"On the linear third-order differential equation" Springer Berlin Heidelberg, 1999. Solving Third Order Linear Differential Equations in Terms of Second Order Equations Mark van Hoeij

order polonium has always one real solution. Thus, derivation of the leading equation (results of the ode) is reduced into quadratic equation and thus the same situation exist.

$$s^3 + a_1\, s^2 + a_2\, s + a_3 = 0 \tag{A.92}$$

The solution is

$$s_1 = -\frac{1}{3}a_1 + (S + T) \tag{A.93}$$

$$s_2 = -\frac{1}{3}a_1 - \frac{1}{2}(S + T) + \frac{1}{2}i\sqrt{3}(S - T) \tag{A.94}$$

and

$$s_3 = -\frac{1}{3}a_1 - \frac{1}{2}(S + T) - \frac{1}{2}i\sqrt{3}(S - T) \tag{A.95}$$

Where

$$S = \sqrt[3]{R + \sqrt{D}}, \tag{A.96}$$

$$T = \sqrt[3]{R - \sqrt{D}} \tag{A.97}$$

and where the D is defined as

$$D = Q^3 + R^2 \tag{A.98}$$

and where the definitions of Q and R are

$$Q = \frac{3a_2 - {a_1}^2}{9} \tag{A.99}$$

and

$$R = \frac{9a_1a_2 - 27a_3 - 2{a_1}^3}{54} \tag{A.100}$$

Only three roots can exist for the Mach angle, θ. From a mathematical point of view, if $D > 0$, one root is real and two roots are complex. For the case $D = 0$, all the roots are real and at least two are identical. In the last case where $D < 0$, all the roots are real and unequal.

When the characteristic equation solution has three different real roots the solution of the differential equation is

$$u = c_1\, e^{s_1\, t} + c_2\, e^{s_2\, t} + c_3\, e^{s_3\, t} \tag{A.101}$$

In the case the solution to the characteristic has two identical real roots

$$u = (c_1 + c_2\, t)\, e^{s_1\, t} + c_3\, e^{s_2\, t} \tag{A.102}$$

Similarly derivations for the case of three identical real roots. For the case of only one real root, the solution is

$$u = (c_1 \sin b_1 + c_2 \cos b_1)\, e^{a_1 t} + c_3 e^{s_3 t} \tag{A.103}$$

Where a_1 is the real part of the complex root and b_1 imaginary part of the root.

A.2.7 Forth and Higher Order ODE

The ODE and partial differential equations (PDE) can be of any integer order. Sometimes the ODE is fourth order or higher the general solution is based in idea that equation is reduced into a lower order. Generally, for constant coefficients ODE can be transformed into multiplication of smaller order linear operations. For example, the equation

$$\frac{d^4 u}{dt^4} - u = 0 \Longrightarrow \left(\frac{d^4}{dt^4} - 1\right) u = 0 \tag{A.104}$$

can be written as combination of

$$\left(\frac{d^2}{dt^2} - 1\right)\left(\frac{d^2}{dt^2} + 1\right) u = 0 \quad \text{or} \quad \left(\frac{d^2}{dt^2} + 1\right)\left(\frac{d^2}{dt^2} - 1\right) u = 0 \tag{A.105}$$

The order of operation is irrelevant as shown in equation (A.105). Thus the solution of

$$\left(\frac{d^2}{dt^2} + 1\right) u = 0 \tag{A.106}$$

with the solution of

$$\left(\frac{d^2}{dt^2} - 1\right) u = 0 \tag{A.107}$$

are the solutions of (A.104). The solution of equation (A.106) and equation (A.107) was discussed earlier.

The general procedure is based on the above concept but is some what simpler. Inserting $e^{s\,t}$ into the ODE

$$a_n\, u^{(n)} + a_{n-1}\, u^{(n-1)} + a_{n-2}\, u^{(n-2)} + \cdots + a_1\, u' + a_0\, u = 0 \tag{A.108}$$

yields characteristic equation

$$a_n\, s^n + a_{n-1}\, s^{n-1} + a_{n-2}\, s^{n-2} + \cdots + a_1\, s + a_0 = 0 \tag{A.109}$$

If The Solution of Characteristic Equation	**The Solution of Differential Equation Is**
all roots are real and different e.g. $s_1 \neq s_2 \neq s_3 \neq s_4 \cdots \neq s_n$	$u = c_1\, e^{s_1 t} + c_2\, e^{s_2 t} + \cdots + c_n\, e^{s_n t}$
all roots are real but some are identical e.g. $s_1 = s_2 = \cdots = s_k$ and some different e.g. $s_{k+1} \neq s_{k+2} \neq s_{k+3} \cdots \neq s_n$	$u = \left(c_1 + c_2\, t + \cdots + c_k\, t^{k-1}\right)\, e^{s_1 t} +$ $c_{k+1}\, e^{s_{k+1} t} + c_{k+2}\, e^{s_{k+2} t} + \cdots + c_n\, e^{s_n t}$
$k/2$ roots, are pairs of conjugate complex numbers of $s_i = a_i \pm b_i$ and some real and different e.g. $s_{k+1} \neq s_{k+2} \neq s_{k+3} \cdots \neq s_n$	$u = (\cos(b_1\, t) + \sin(b_1\, t))\, e^{a_1 t} +$ $\cdots + (\cos(b_i\, t) + \sin(b_i\, t))\, e^{a_i t} +$ $\cdots + (\cos(b_k\, t) + \sin(b_k\, t))\, e^{a_k t} +$ $c_{k+1}\, e^{s_{k+1} t} + c_{k+2}\, e^{s_{k+2} t} + \cdots + c_n\, e^{s_n t}$
$k/2$ roots, are pairs of conjugate complex numbers of $s_i = a_i \pm b_i$, ℓ roots are similar and some real and different e.g. $s_{k+1} \neq s_{k+2} \neq s_{k+3} \cdots \neq s_n$	$u = (\cos(b_1\, t) + \sin(b_1\, t))\, e^{a_1 t} +$ $\cdots + (\cos(b_i\, t) + \sin(b_i\, t))\, e^{a_i t} +$ $\cdots + (\cos(b_k\, t) + \sin(b_k\, t))\, e^{a_k t} +$ $\left(c_{k+1} + c_{k+2}\, t + \cdots + c_{k+\ell}\, t^{\ell-1}\right)\, e^{s_{k+1} t} +$ $c_{k+2}\, e^{s_{k+2} t} + c_{k+3}\, e^{s_{k+3} t} + \cdots + c_n\, e^{s_n t}$

Example A.10:
Solve the fifth order ODE

$$\frac{d^5u}{dt^5} - 11\,\frac{d^4u}{dt^4} + 57\,\frac{d^3u}{dt^3} - 149\,\frac{d^2u}{dt^2} + 192\,\frac{du}{dt} - 90\,u = 0 \tag{1.X.a}$$

<u>Solution</u>

The characteristic equation is

$$s^5 - 11\,s^4 + 57\,s^3 - 149\,s^2 + 192\,s - 90 = 0 \tag{1.X.b}$$

With the roots of the equation (1.X.b) (these roots can be found using numerical methods or Descartes' Rule) are

$$\begin{aligned} s_{1,2} &= 3 \pm 3\,i \\ s_{3,4} &= 2 \pm i \\ s_5 &= 1 \end{aligned} \tag{1.X.c}$$

The roots are two pairs of complex numbers and one real number. Thus the solution is

$$u = c_1\, e^t + e^{2\,t}\,\left(c_2\,\sin(t) + c_3\,\cos(t)\right) + e^{3\,t}\,\left(c_4\,\sin(3\,t) + c_5\,\cos(3\,t)\right) \tag{1.X.d}$$

End Solution

A.2.8 A general Form of the Homogeneous Equation

The homogeneous equation can be generalized to

$$k_0\, t^n \frac{d^n u}{dt^n} + k_1\, t^{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \cdots + k_{n-1}\, t \frac{du}{dt} + k_n\, u = a\, x \tag{A.110}$$

To be continue

A.3 Partial Differential Equations

Partial Differential Equations (PDE) are differential equations which include function includes the partial derivatives of two or more variables. Example of such equation is

$$F(u_t, u_x, \ldots) = 0 \tag{A.111}$$

Where subscripts refers to derivative based on it. For example, $u_x = \frac{\partial u}{\partial x}$. Note that partial derivative also include mix of derivatives such as $u_x y$. As one might expect PDE are harder to solve.

Many situations in fluid mechanics can be described by PDE equations. Generally, the PDE solution is done by transforming the PDE to one or more ODE. Partial differential equations are categorized by the order of highest derivative. The nature of the solution is based whether the equation is elliptic parabolic and hyperbolic. Normally, this characterization is done for for second order. However, sometimes similar definition can be applied for other order. The physical meaning of the these definition is that these equations have different characterizations. The solution of elliptic equations depends on the boundary conditions The solution of parabolic equations depends on the boundary conditions but as well on the initial conditions. The hyperbolic equations are associated with method of characteristics because physical situations depends only on the initial conditions. The meaning for initial conditions is that of solution depends on some early points of the flow (the solution). The general second-order PDE in two independent variables has the form

$$a_{xx}\, u_{xx} + 2a_{xy}\, u_{xy} + a_{yy}\, u_{yy} + \cdots = 0 \tag{A.112}$$

The coefficients a_{xx}, a_{xy}, a_{yy} might depend upon "x" and "y". Equation (A.112) is similar to the equations for a conic geometry:

$$a_{xx}\, x^2 + a_{xy}\, x\, y + a_{yy}\, y^2 + \cdots = 0 \tag{A.113}$$

In the same manner that conic geometry equations are classified are based on the discriminant $a_{xy}^2 - 4\, a_{xx}\, a_{yy}$, the same can be done for a second-order PDE. The discriminant can be function of the x and y and thus can change sign and thus the characteristic of the equation. Generally, when the discriminant is zero the equation are called parabolic. One example of such equation is heat equation. When the discriminant is larger then zero the equation is referred as hyperbolic equations. In fluid mechanics this kind equation appear in supersonic flow or in supper critical flow in open channel flow. The equations that not mentioned above are elliptic which appear in ideal flow and subsonic flow and sub critical open channel flow.

A.3.1 First-order equations

First order equation can be written as

$$u = a_x \frac{\partial u}{\partial x} + a_y \frac{\partial u}{\partial x} + \dots \tag{A.114}$$

The interpretation the equation characteristic is complicated. However, the physics dictates this character and will be used in the book.

An example of first order equation is

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{A.115}$$

The solution is assume to be $u = Y(y)\,X(x)$ and substitute into the (A.115) results in

$$Y(y)\,\frac{\partial X(x)}{\partial x} + X(x)\,\frac{\partial Y(y)}{\partial y} = 0 \tag{A.116}$$

Rearranging equation (A.116) yields

$$\frac{1}{X(x)}\,\frac{\partial X(x)}{\partial x} + \frac{1}{Y(y)}\,\frac{\partial Y(y)}{\partial y} = 0 \tag{A.117}$$

A possible way the equation (A.117) can exist is that these two term equal to a constant. Is it possible that these terms not equal to a constant? The answer is no if the assumption of the solution is correct. If it turned that assumption is wrong the ratio is not constant. Hence, the constant is denoted as λ and with this definition the PDE is reduced into two ODE. The first equation is X function

$$\frac{1}{X(x)}\,\frac{\partial X(x)}{\partial x} = \lambda \tag{A.118}$$

The second ODE is for Y

$$\frac{1}{Y(y)}\,\frac{\partial Y(y)}{\partial y} = -\lambda \tag{A.119}$$

Equations (A.119) and (A.118) are ODE that can be solved with the methods described before for certain boundary condition.

A.4 Trigonometry

These trigonometrical identities were set up by Keone Hon with slight modification

1. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
2. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
3. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
4. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
5. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$
6. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
2. $\cos 2\alpha = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
3. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$
4. $\sin\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ (determine whether it is $+$ or - by finding the quadrant that $\dfrac{\alpha}{2}$ lies in)
5. $\cos\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ (same as above)
6. $\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1+\cos\alpha}$

for formulas 3-6, consider the triangle with sides of length a, b, and c, and opposite angles α, β, and γ, respectively

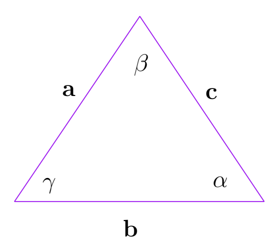

Fig. -A.7. The tringle angles sides.

1. $\sin^2\alpha = \dfrac{1-2\cos(2\alpha)}{2}$
2. $\cos^2\alpha = \dfrac{1+2\cos(2\alpha)}{2}$
3. $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$ (Law of Sines)
4. $c^2 = a^2 + b^2 - 2\,a\,b\cos\gamma$ (Law of Cosines)
5. Area of triangle $= \frac{1}{2}a\,b\sin\gamma$
6. Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$ (Heron's Formula)