# Note:

CHAPTER 9: DIFFERENTIAL

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This chapter is part of the textbook:

**“Basics of Fluid Mechanics”**

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# THE LIST OF THE AVAILABLE BOOKS IN POTTO PROJECT

| **Project Name** | **Progress** | **Remarks** | **Version** | **Avail-ability for Public Download** |
|---|---|---|---|---|
| Compressible Flow | beta | | 0.4.8.6 | ✔ |
| Gas Dynamics Tables | final | World biggest | 1.2 | ✔ |
| Die Casting | alpha | | 0.1.2 | ✔ |
| Dynamics | NSY | | 0.0.0 | ✖ |
| Fluid Mechanics | beta | | 0.2.9 | ✔ |
| Heat Transfer | NSY | Based on Eckert | 0.0.0 | ✖ |
| Mechanics | NSY | | 0.0.0 | ✖ |
| Open Channel Flow | NSY | | 0.0.0 | ✖ |
| Statics | early alpha | first chapter | 0.0.1 | ✖ |
| Strength of Material | NSY | | 0.0.0 | ✖ |
| Thermodynamics | early alpha | | 0.0.01 | ✖ |
| Two/Multi phases flow | NSY | Tel-Aviv's notes | 0.0.0 | ✖ |

NSY = Not Started Yet

## Part II

# Differential Analysis

# CHAPTER 8

## Differential Analysis

### 8.1 Introduction

The integral analysis has a limited accuracy, which leads to a different approach of differential analysis. The differential analysis allows the flow field investigation in greater detail. In differential analysis, the emphasis is on infinitesimal scale and thus the analysis provides better accuracy[1]. This analysis leads to partial differential equations which are referred to as the Navier-Stokes equations. These equations are named after Claude–Louis Navier–Marie and George Gabriel Stokes. Like many equations they were independently derived by several people. First these equations were derived by Claude–Louis–Marie Navier as it is known in 1827. As usual Simon-Denis Poisson independently, as he done to many other equations or conditions, derived these equations in 1831 for the same arguments as Navier. The foundations for their arguments or motivations are based on a molecular view of how stresses are exerted between fluid layers. Barré de Saint Venant (1843) and George Gabriel Stokes (1845) derived these equation based on the relationship between stress and rate–of–strain (this approach is presented in this book).

Navier-Stokes equations are non–linear and there are more than one possible solution in many cases (if not most cases) e.g. the solution is not unique. A discussion about the "regular" solution is present and a brief discussion about limitations when the solution is applicable. Later in the Chapters on Real Fluid and Turbulence, with a presentation of the "non–regular" solutions will be presented with the associated issues of stability. However even for the "regular" solution the mathematics is very complex. One of the approaches is to reduce the equations by eliminating the viscosity effects. The equations without the viscosity effects are referred to as the ideal flow equations (Euler Equations) which will be discussed in the next chapter. The concepts

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[1] Which can be view as complementary analysis to the integral analysis.

of the Add Mass and the Add Force, which are easier to discuss when the viscosity is ignored, and will be presented in the Ideal Flow chapter. It has to be pointed out that the Add Mass and Add Force appear regardless to the viscosity. Historically, complexity of the equations, on one hand, leads to approximations and consequently to the ideal flow approximation (equations) and on the other hand experimental solutions of Navier–Stokes equations. The connection between these two ideas or fields was done via introduction of the boundary layer theory by Prandtl which will be discussed as well.

Even for simple situations, there are cases when complying with the boundary conditions leads to a discontinuity (shock or choked flow). These equations cannot satisfy the boundary conditions in other cases and in way the fluid pushes the boundary condition(s) further downstream (choked flow). These issues are discussed in Open Channel Flow and Compressible Flow chapters. Sometimes, the boundary conditions create instability which alters the boundary conditions itself which is known as Interfacial instability. The choked flow is associated with a single phase flow (even the double choked flow) while the Interfacial instability associated with the Multi–Phase flow. This phenomenon is presented in Multi–phase chapter and briefly discussed in this chapter.

## 8.2 Mass Conservation

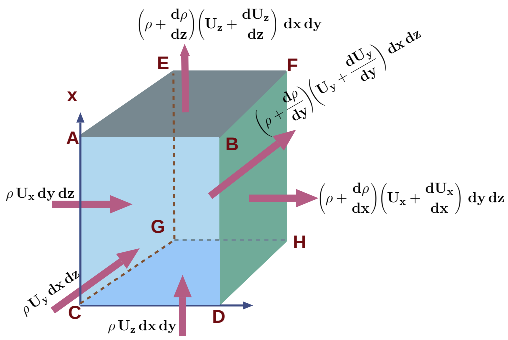

*Fig. -8.1. The mass balance on the infinitesimal control volume.*

Fluid flows into and from a three dimensional infinitesimal control volume depicted in Figure 8.1. At a specific time this control volume can be viewed as a system. The mass conservation for this infinitesimal small system is zero thus

$$\frac{D}{Dt}\int_V \rho dV = 0 \tag{8.1}$$

However for a control volume using Reynolds Transport Theorem (RTT), the following can be written

$$\frac{D}{Dt}\int_V \rho dV = \frac{d}{dt}\int_V \rho dV + \int_A U_{rn}\,\rho\,dA = 0 \tag{8.2}$$

For a constant control volume, the derivative can enter into the integral (see also for the divergence theorem in the appendix A.1.2) on the right hand side and hence

$$\overbrace{\int_V \frac{d\rho}{dt}\,dV}^{\frac{d\rho}{dt}\,dV} + \int_A U_{rn}\,\rho\,dA = 0 \tag{8.3}$$

The first term in equation (8.3) for the infinitesimal volume is expressed, neglecting higher order derivatives, as

$$\int_V \frac{d\rho}{dt}\,dV = \frac{d\rho}{dt}\overbrace{dx\,dy\,dz}^{dV} + \overbrace{f\left(\frac{d^2\rho}{dt^2}\right) + \cdots}^{\sim 0} \tag{8.4}$$

The second term in the LHS of equation (8.2) is expressed[2] as

$$\begin{aligned}\int_A U_{rn}\,\rho\,dA = {} & \overbrace{dy\,dz}^{dA_{yz}}\left[\left.(\rho\,U_x)\right|_x - \left.(\rho\,U_x)\right|_{x+dx}\right] + \\ & \overbrace{dx\,dz}^{dA_{xz}}\left[\left.(\rho\,U_y)\right|_y - \left.(\rho\,U_y)\right|_{y+dy}\right] + \overbrace{dx\,dy}^{dA_{xz}}\left[\left.(\rho\,U_z)\right|_z - \left.(\rho\,U_z)\right|_{z+dz}\right]\end{aligned} \tag{8.5}$$

The difference between point $x$ and $x+dx$ can be obtained by developing Taylor series as

$$\left.(\rho\,U_x)\right|_{x+dx} = \left.(\rho\,U_x)\right|_x + \left.\frac{\partial\,(\rho\,U_x)}{\partial x}\right|_x dx \tag{8.6}$$

The same can be said for the $y$ and $z$ coordinates. It also can be noticed that, for example, the operation, in the $x$ coordinate, produces additional $dx$ thus a infinitesimal volume element $dV$ is obtained for all directions. The combination can be divided by $dx\,dy\,dz$ and simplified by using the definition of the partial derivative in the regular process to be

$$\int_A U_{rn}\,\rho\,dA = -\left[\frac{\partial(\rho\,U_x)}{\partial x} + \frac{\partial(\rho\,U_y)}{\partial y} + \frac{\partial(\rho\,U_z)}{\partial z}\right] \tag{8.7}$$

Combining the first term with the second term results in the continuity equation in Cartesian coordinates as

**Continuity in Cartesian Coordinates**

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho\,U_x}{\partial x} + \frac{\partial\rho\,U_y}{\partial y} + \frac{\partial\rho\,U_z}{\partial z} = 0 \tag{8.8}$$

### Cylindrical Coordinates

The same equation can be derived in cylindrical coordinates. The net mass change, as depicted in Figure 8.2, in the control volume is

$$d\,\dot{m} = \frac{\partial\rho}{\partial t}\overbrace{dr\,dz\,r\,d\theta}^{dv} \tag{8.9}$$

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[2] Note that sometime the notation $dA_{yz}$ also refers to $dA_x$.

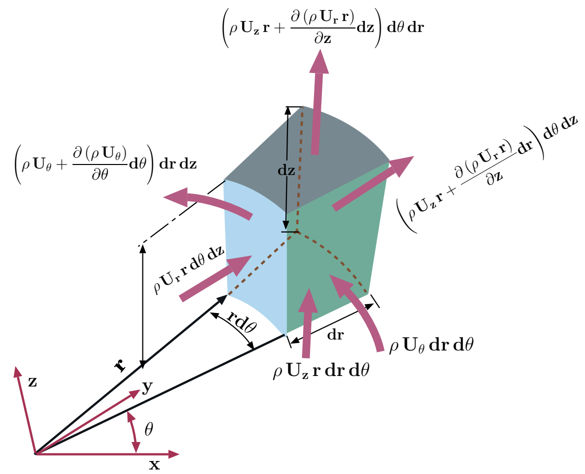

*Fig. -8.2. The mass conservation in cylindrical coordinates.*

The net mass flow out or in the $\widehat{\mathbf{r}}$ direction has an additional term which is the area change compared to the Cartesian coordinates. This change creates a different differential equation with additional complications. The change is

$$\begin{pmatrix}\text{flux in } r\\ \text{direction}\end{pmatrix}=d\theta\,dz\left(r\,\rho\,U_r-\left(r\,\rho\,U_r+\frac{\partial\rho\,U_r\,r}{\partial r}dr\right)\right)\tag{8.10}$$

The net flux in the $r$ direction is then

$$\begin{pmatrix}\text{net flux in the}\\ r\text{ direction}\end{pmatrix}=d\theta\,dz\,\frac{\partial\rho\,U_r\,r}{\partial r}dr\tag{8.11}$$

Note[3] that the $r$ is still inside the derivative since it is a function of $r$, e.g. the change of $r$ with $r$. In a similar fashion, the net flux in the $z$ coordinate be written as

$$\text{net flux in } z \text{ direction}=r\,d\theta\,dr\,\frac{\partial\left(\rho\,U_z\right)}{\partial z}dz\tag{8.12}$$

The net change in the $\theta$ direction is then

$$\text{net flux in } \theta \text{ direction}=dr\,dz\,\frac{\partial\rho\,U_\theta}{\partial\theta}d\theta\tag{8.13}$$

Combining equations (8.11)–(8.13) and dividing by infinitesimal control volume, $dr\;r\,d\theta\,dz$, results in

$$\begin{pmatrix}\text{total}\\ \text{net flux}\end{pmatrix}=-\left(\frac{1}{r}\frac{\partial\left(\rho\,U_r\,r\right)}{\partial r}+\frac{\partial\rho\,U_z\,r}{\partial z}+\frac{\partial\rho\,U_\theta}{\partial\theta}\right)\tag{8.14}$$

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[3] The mass flow is $\rho\,U_r\,r\,d\theta\,dz$ at $r$ point. Expansion to Taylor serious $\rho\,U_r\,r\,d\theta\,dz|_{r+dr}$ is obtained by the regular procedure. The mass flow at $r+dr$ is $\rho\,U_r\,r\,d\theta\,dz|_r+d/dr\left(\rho\,U_r\,r\,d\theta\,dz\right)\,dr+\cdots$. Hence, the $r$ is "trapped" in the derivative.

Combining equation (8.14) with the change in the control volume (8.9) divided by infinitesimal control volume, $dr\, r\, d\theta\, dz$ yields

Continuity in Cylindrical Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \left(r\, \rho\, U_r\right)}{\partial r} + \frac{1}{r}\frac{\partial \rho\, U_\theta}{\partial \theta} + \frac{\partial \rho\, U_z}{\partial z} = 0 \tag{8.15}$$

Carrying similar operations for the spherical coordinates, the continuity equation becomes

Continuity in Spherical Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial \left(r^2\, \rho\, U_r\right)}{\partial r} + \frac{1}{r\, \sin\theta}\frac{\partial \left(\rho\, U_\theta\, \sin\theta\right)}{\partial \theta} + \frac{1}{r\, \sin\theta}\frac{\partial \rho\, U_\phi}{\partial z} = 0 \tag{8.16}$$

The continuity equations (8.8), (8.15) and (8.16) can be expressed in different coordinates. It can be noticed that the second part of these equations is the divergence (see the Appendix A.1.2 page 366). Hence, the continuity equation can be written in a general vector form as

Continuity Equation

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho\, \boldsymbol{U}\right) = 0} \tag{8.17}$$

*Advance material can be skipped*

The mass equation can be written in index notation for Cartesian coordinates. The index notation really does not add much to the scientific understanding. However, this writing reduce the amount of writing and potentially can help the thinking about the problem or situation in more conceptional way. The mass equation (see in the appendix for more information on the index notation) written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho\, U\right)_i}{\partial x_i} = 0 \tag{8.18}$$

Where $i$ is is of the $i$, $j$, and $k$[4]. Compare to equation (8.8). Again remember that the meaning of repeated index is summation.

*End Advance material*

The use of these equations is normally combined with other equations (momentum and or energy equations). There are very few cases where this equation is used on its own merit. For academic purposes, several examples are constructed here.

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[4] notice the irony the second $i$ is the dirction and first $i$ is for any one of direction x(i), y(j), and z(k).

### 8.2.1 Mass Conservation Examples

Example 8.1:

*A layer of liquid has an initial height of $H_0$ with an uniform temperature of $T_0$. At time, $t_0$, the upper surface is exposed to temperature $T_1$ (see Figure 8.3). Assume that the actual temperature is exponentially approaches to a linear temperature profile as depicted in Figure 8.3. The density is a function of the temperature according to*

$$\frac{T-T_0}{T_1-T_0}=\alpha\left(\frac{\rho-\rho_0}{\rho_1-\rho_0}\right) \tag{8.I.a}$$

*where $\rho_1$ is the density at the surface and where $\rho_0$ is the density at the bottom. Assume that the velocity is only a function of the $y$ coordinate. Calculates the velocity of the liquid. Assume that the velocity at the lower boundary is zero at all times. Neglect the mutual dependency of the temperature and the height.*

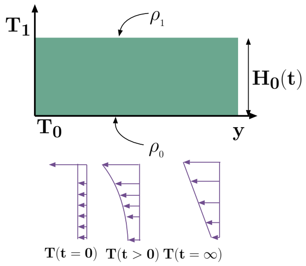

*Fig. -8.3. Mass flow due to temperature difference for example 8.1*

SOLUTION

The situation is unsteady state thus the unsteady state and one dimensional continuity equation has to be used which is

$$\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho U_y\right)}{\partial y}=0 \tag{8.I.b}$$

with the boundary condition of zero velocity at the lower surface $U_y(y=0)=0$. The expression that connects the temperature with the space for the final temperature as

$$\frac{T-T_0}{T_1-T_0}=\alpha\,\frac{H_0-y}{H_0} \tag{8.I.c}$$

The exponential decay is $\left(1-e^{-\beta\,t}\right)$ and thus the combination (with equation (8.I.a)) is

$$\frac{\rho-\rho_0}{\rho_1-\rho_0}=\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right) \tag{8.I.d}$$

Equation (8.I.d) relates the temperature with the time and the location was given in the question (it is not the solution of any model). It can be noticed that the height $H_0$ is a function of time. For this question, it is treated as a constant. Substituting the density, $\rho$, as a function of time into the governing equation (8.I.b) results in

$$\overbrace{\alpha\,\beta\left(\frac{H_0-y}{H_0}\right)e^{-\beta\,t}}^{\frac{\partial\rho}{\partial t}}+\overbrace{\frac{\partial\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right)}{\partial y}}^{\frac{\partial\rho\,U_y}{\partial y}}=0 \tag{8.I.e}$$

Equation (8.I.e) is first order ODE with the boundary condition $U_y(y=0)=0$ which can be arranged as

$$\frac{\partial\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right)}{\partial y} = -\alpha\,\beta\left(\frac{H_0-y}{H_0}\right)e^{-\beta\,t} \tag{8.I.f}$$

$U_y$ is a function of the time but not $y$. Equation (8.I.f) holds for any time and thus, it can be treated for the solution of equation (8.I.f) as a constant[5]. Hence, the integration with respect to $y$ yields

$$\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right) = -\alpha\,\beta\left(\frac{2\,H_0-y}{2\,H_0}\right)e^{-\beta\,t}y+c \tag{8.I.g}$$

Utilizing the boundary condition $U_y(y=0)=0$ yields

$$\left(U_y\,\alpha\,\frac{H_0-y}{H_0}\left(1-e^{-\beta\,t}\right)\right) = -\alpha\,\beta\left(\frac{2\,H_0-y}{2\,H_0}\right)e^{-\beta\,t}\left(y-1\right) \tag{8.I.h}$$

or the velocity is

$$U_y = \beta\left(\frac{2\,H_0-y}{2\,\left(H_0-y\right)}\right)\frac{e^{-\beta\,t}}{\left(1-e^{-\beta\,t}\right)}\left(1-y\right) \tag{8.I.i}$$

It can be noticed that indeed the velocity is a function of the time and space $y$.

End Solution

## 8.2.2 Simplified Continuity Equation

A simplified equation can be obtained for a steady state in which the transient term is eliminated as (in a vector form)

$$\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{U}\right)=0 \tag{8.19}$$

If the fluid is incompressible then the governing equation is a volume conservation as

$$\boldsymbol{\nabla}\cdot\boldsymbol{U}=0 \tag{8.20}$$

Note that this equation appropriate only for a single phase case.

Example 8.2:

*In many coating processes a thin film is created by a continuous process in which liquid injected into a moving belt which carries the material out as exhibited in Figure 8.4.*

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[5] Since the time can be treated as a constant for $y$ integration.

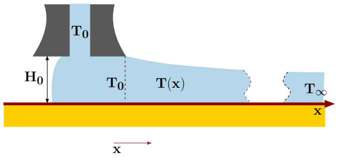

*Fig. -8.4. Mass flow in coating process for example 8.2.*

*The temperature and mass transfer taking place which reduces (or increases) the thickness of the film. For this example, assume that no mass transfer occurs or can be neglected and the main mechanism is heat transfer. Assume that the film temperature is only a function of the distance from the extraction point. Calculate the film velocity field if the density is a function of the temperature. The relationship between the density and the temperature is linear as*

$$\frac{\rho-\rho_\infty}{\rho_0-\rho_\infty}=\alpha\left(\frac{T-T_\infty}{T_0-T_\infty}\right) \tag{8.II.a}$$

*State your assumptions.*

Solution

This problem is somewhat similar to Example 8.1[6], however it can be considered as steady state. At any point the governing equation in coordinate system that moving with the belt is

$$\frac{\partial\left(\rho\,U_x\right)}{\partial x}+\frac{\partial\left(\rho\,U_y\right)}{\partial y}=0 \tag{8.II.b}$$

At first, it can be assumed that the material moves with the belt in the $x$ direction in the same velocity. This assumption is consistent with the first solution (no stability issues). If the frame of reference was moving with the belt then there is only velocity component in the $y$ direction[7]. Hence equation (8.II.b) can be written as

$$U_x\,\frac{\partial\rho}{\partial x}=-\frac{\partial\left(\rho\,U_y\right)}{\partial y} \tag{8.II.c}$$

Where $U_x$ is the belt velocity.

See the resembles to equation (8.I.b). The solution is similar to the previous Example 8.1 for a general function $T=F(x)$.

$$\frac{\partial\rho}{\partial x}=\frac{\alpha}{U_x}\,\frac{\partial F(x)}{\partial x}\left(\rho_0-\rho_\infty\right) \tag{8.II.d}$$

Substituting this relationship in equation (8.II.d) into the governing equation results in

$$\frac{\partial U_y\,\rho}{\partial y}=\frac{\alpha}{U_x}\,\frac{\partial F(x)}{\partial x}\left(\rho_0-\rho_\infty\right) \tag{8.II.e}$$

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[6] The presentation of one dimension time dependent problem to two dimensions problems can be traced to heat and mass transfer problems. One of the early pioneers who suggest this idea is Higbie which Higbie's equation named after him. Higbie's idea which was rejected by the scientific establishment. He spend the rest of his life to proof it and ending in a suicide. On personal note, this author Master thesis is extension Higbie's equation.

[7] In reality this assumption is correct only in a certain range. However, the discussion about this point is beyond the scope of this section.

The density is expressed by equation (8.II.a) and thus

$$U_y = \frac{\alpha}{\rho\, U_x} \frac{\partial F(x)}{\partial x} \left(\rho_0 - \rho_\infty\right) y + c \tag{8.II.f}$$

Notice that $\rho$ could "come" out of the derivative (why?) and move into the RHS. Applying the boundary condition $U_y(t=0)=0$ results in

$$U_y = \frac{\alpha}{\rho(x)\, U_x} \frac{\partial F(x)}{\partial x} \left(\rho_0 - \rho_\infty\right) y \tag{8.II.g}$$

End Solution

## Example 8.3:

*The velocity in a two dimensional field is assumed to be in a steady state. Assume that the density is constant and calculate the vertical velocity ($y$ component) for the following $x$ velocity component.*

$$U_x = a\,x^2 + b\,y^2 \tag{8.III.a}$$

*Next, assume the density is also a function of the location in the form of*

$$\rho = m\,e^{x+y} \tag{8.III.b}$$

*Where $m$ is constant. Calculate the velocity field in this case.*

Solution

The flow field must comply with the mass conservation (8.20) thus

$$2\,a\,x + \frac{\partial U_y}{\partial y} = 0 \tag{8.III.c}$$

Equation (8.III.c) is an ODE with constant coefficients. It can be noted that $x$ should be treated as a constant parameter for the $y$ coordinate integration. Thus,

$$U_y = -\int 2\,a\,x + f(x) = -2\,x\,y + f(x) \tag{8.III.d}$$

The integration constant in this case is not really a constant but rather an arbitrary function of $x$. Notice the symmetry of the situation. The velocity, $U_x$ has also arbitrary function in the $y$ component.

For the second part equation (8.19) is applicable and used as

$$\frac{\partial \left(a\,x^2 + b\,y^2\right)\left(m\,e^{x+y}\right)}{\partial x} + \frac{\partial\, U_y \left(m\,e^{x+y}\right)}{\partial y} = 0 \tag{8.III.e}$$

Taking the derivative of the first term while moving the second part to the other side results in

$$a\left(2\,x+x^2+\frac{b}{a}\,y^2\right)\,e^{x+y}=-\left(e^{x+y}\right)\left(\frac{\partial\,U_y}{\partial y}+U_y\right) \tag{8.III.f}$$

The exponent can be canceled to further simplify the equation (8.III.f) and switching sides to be

$$\left(\frac{\partial\,U_y}{\partial y}+U_y\right)=-a\left(2\,x+x^2+\frac{b}{a}\,y^2\right) \tag{8.III.g}$$

Equation (8.III.g) is a first order ODE that can be solved by combination of the homogeneous solution with the private solution (see for an explanation in the Appendix). The homogeneous equation is

$$\left(\frac{\partial\,U_y}{\partial y}+U_y\right)=0 \tag{8.III.h}$$

The solution for (8.III.h) is $U_y=c\,e^{-y}$ (see for an explanation in the appendix). The private solution is

$$U_y|_{private}=\left(-b\,\left(y^2-2\,y+2\right)-a\,x^2-2\,a\,x\right) \tag{8.III.i}$$

The total solution is

$$U_y=c\,e^{-y}+\left(-b\,\left(y^2-2\,y+2\right)-a\,x^2-2\,a\,x\right) \tag{8.III.j}$$

End Solution

## Example 8.4:

*Can the following velocities co-exist*

$$U_x=\left(x\,t\right)^2\,z \qquad U_y=\left(x\,t\right)+\left(y\,t\right)+\left(z\,t\right) \qquad U_z=\left(x\,t\right)+\left(y\,t\right)+\left(z\,t\right) \tag{8.IV.a}$$

*in the flow field. Is the flow is incompressible? Is the flow in a steady state condition?*

Solution

Whether the solution is in a steady state or not can be observed from whether the velocity contains time component. Thus, this flow field is not steady state since it contains time componnet. This continuity equation is checked if the flow incompressible (constant density). The derivative of each componnet are

$$\frac{\partial U_x}{\partial x}=t^2\,z \qquad \frac{\partial U_y}{\partial y}=t \qquad \frac{\partial U_z}{\partial z}=t \tag{8.IV.b}$$

Hence the gradient or the combination of these derivatives is

$$\nabla\boldsymbol{U}=t^2\,z+2\,t \tag{8.IV.c}$$

The divergence isn't zero thus this flow, if it exist, must be compressible flow. This flow can exist only for a limit time since over time the divergence is unbounded (a source must exist).

End Solution

## Example 8.5:

*Find the density as a function of the time for a given one dimensional flow with* $U_x = x\, e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right)$*. The initial density is* $\rho(t=0) = \rho_0$*.*

SOLUTION

This problem is one dimensional unsteady state and for a compressible substance. Hence, the mass conservation is reduced only for one dimensional form as

$$\frac{\partial\rho}{\partial t} + \frac{\partial\left(U_x\,\rho\right)}{\partial x} = 0 \tag{8.V.a}$$

Mathematically speaking, this kind of presentation is possible. However physically there are velocity components in $y$ and $z$ directions. In this problem, these physical components are ignored for academic reasons. Equation (8.V.a) is first order partial differential equation which can be converted to an ordinary differential equations when the velocity component, $U_x$, is substituted. Using,

$$\frac{\partial U_x}{\partial x} = e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right) \tag{8.V.b}$$

Substituting equation (8.V.b) into equation (8.V.a) and noticing that the density, $\rho$, is a function of $x$ results of

$$\frac{\partial\rho}{\partial t} = -\rho\,x\,e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right) - \frac{\partial\rho}{\partial x}\,e^{5\,\alpha\,y}\,\left(\cos\left(\alpha\,t\right)\right) \tag{8.V.c}$$

Equation (8.V.c) can be separated to yield

$$\overbrace{\frac{1}{\cos\left(\alpha\,t\right)}\frac{\partial\rho}{\partial t}}^{f(t)} = \overbrace{-\rho\,x\,e^{5\,\alpha\,y} - \frac{\partial\rho}{\partial x}\,e^{5\,\alpha\,y}}^{f(y)} \tag{8.V.d}$$

A possible solution is when the left and the right hand sides are equal to a constant. In that case the left hand side is

$$\frac{1}{\cos\left(\alpha\,t\right)}\frac{\partial\rho}{\partial t} = c_1 \tag{8.V.e}$$

The solution of equation (8.V.e) is reduced to ODE and its solution is

$$\rho = \frac{c_1\,sin\left(\alpha\,t\right)}{\alpha} + c_2 \tag{8.V.f}$$

The same can be done for the right hand side as

$$\rho\, x\, e^{5\,\alpha\, y} + \frac{\partial\rho}{\partial x}\, e^{5\,\alpha\, y} = c_1 \tag{8.V.g}$$

The term $e^{5\,\alpha\, y}$ is always positive, real value, and independent of $y$ thus equation (8.V.g) becomes

$$\rho\, x + \frac{\partial\rho}{\partial x} = \frac{c_1}{e^{5\,\alpha\, y}} = c_3 \tag{8.V.h}$$

Equation (8.V.h) is a constant coefficients first order ODE which its solution discussed extensively in the appendix. The solution of (8.V.h) is given by

$$\rho = e^{-\frac{x^2}{2}} \left( c - \frac{\overbrace{\sqrt{\pi}\, i\, c_3\, erf\left(\frac{i\,x}{\sqrt{2}}\right)}^{\text{impossible solution}}}{\sqrt{2}} \right) \tag{8.V.i}$$

which indicates that the solution is a complex number thus the constant, $c_3$, must be zero and thus the constant, $c_1$ vanishes as well and the solution contain only the homogeneous part and the private solution is dropped

$$\rho = c_2\, e^{-\frac{x^2}{2}} \tag{8.V.j}$$

The solution is the multiplication of equation (8.V.j) by (8.V.f) transfered to

$$\rho = c_2\, e^{-\frac{x^2}{2}} \left( \frac{c_1\, sin\,(\alpha\, t)}{\alpha} + c_2 \right) \tag{8.V.k}$$

Where the constant, $c_2$, is an arbitrary function of the $y$ coordinate.

End Solution

## 8.3 Conservation of General Quantity

### 8.3.1 Generalization of Mathematical Approach for Derivations

In this section a general approach for the derivations for conservation of any quantity e.g. scalar, vector or tensor, are presented. Suppose that the property $\phi$ is under a study which is a function of the time and location as $\phi(x, y, z, t)$. The total amount of quantity that exist in arbitrary system is

$$\Phi = \int_{sys} \phi\, \rho\, dV \tag{8.21}$$

Where $\Phi$ is the total quantity of the system which has a volume $V$ and a surface area of $A$ which is a function of time. A change with time is

$$\frac{D\Phi}{Dt} = \frac{D}{Dt} \int_{sys} \phi\, \rho\, dV \tag{8.22}$$

Using RTT to change the system to a control volume (see equation (5.33)) yields

$$\frac{D}{Dt}\int_{sys}\phi\,\rho\,dV = \frac{d}{dt}\int_{cv}\phi\,\rho\,dV + \int_{A}\rho\,\phi\boldsymbol{U}\cdot dA \tag{8.23}$$

The last term on the RHS can be converted using the divergence theorem (see the appendix[8]) from a surface integral into a volume integral (alternatively, the volume integral can be changed to the surface integral) as

$$\int_{A}\rho\,\phi\boldsymbol{U}\cdot dA = \int_{V}\nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)dV \tag{8.24}$$

Substituting equation (8.24) into equation (8.23) yields

$$\frac{D}{Dt}\int_{sys}\phi\,\rho\,dV = \frac{d}{dt}\int_{cv}\phi\,\rho\,dV + \int_{cv}\nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)dV \tag{8.25}$$

Since the volume of the control volume remains independent of the time, the derivative can enter into the integral and thus combining the two integrals on the RHS results in

$$\frac{D}{Dt}\int_{sys}\phi\,\rho\,dV = \int_{cv}\left(\frac{d\left(\phi\,\rho\right)}{dt} + \nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)\right)dV \tag{8.26}$$

The definition of equation (8.21) LHS can be changed to simply the derivative of $\Phi$. The integral is carried over arbitrary system. For an infinitesimal control volume the change is

$$\frac{D\,\Phi}{Dt} \cong \left(\frac{d\left(\phi\,\rho\right)}{dt} + \nabla\cdot\left(\rho\,\phi\boldsymbol{U}\right)\right)\overbrace{dx\,dy\,dz}^{dV} \tag{8.27}$$

## 8.3.2 Examples of Several Quantities

### 8.3.2.1 The General Mass Time Derivative

Using $\phi = 1$ is the same as dealing with the mass conservation. In that case $\frac{D\,\Phi}{Dt} = \frac{D\,\rho}{Dt}$ which is equal to zero as

$$\int\left(\frac{d\left(\overbrace{1}^{\phi}\,\rho\right)}{dt} + \nabla\cdot\left(\rho\,\overbrace{1}^{\phi}\,\boldsymbol{U}\right)\right)\overbrace{dx\,dy\,dz}^{dV} = 0 \tag{8.28}$$

---

[8] These integrals are related to RTT. Basically the divergence theorem relates the flow out (or) in and the sum of the all the changes inside the control volume.

Using equation (8.21) leads to

$$\frac{D\rho}{Dt} = 0 \longrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0 \tag{8.29}$$

Equation (8.29) can be rearranged as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{U}\,\nabla \cdot \rho + \rho \nabla \cdot \boldsymbol{U} = 0 \tag{8.30}$$

Equation (8.30) can be further rearranged so derivative of the density is equal the divergence of velocity as

$$\frac{1}{\rho}\left( \overbrace{\frac{\partial \rho}{\partial t} + \boldsymbol{U}\,\nabla \cdot \rho}^{\text{substantial derivative}} \right) = -\nabla \cdot \boldsymbol{U} \tag{8.31}$$

Equation (8.31) relates the density rate of change or the volumetric change to the velocity divergence of the flow field. The term in the bracket LHS is referred in the literature as substantial derivative. The substantial derivative represents the change rate of the density at a point which moves with the fluid.

**Acceleration Direct Derivations**

One of the important points is to find the fluid particles acceleration. A fluid particle velocity is a function of the location and time. Therefore, it can be written that

$$\boldsymbol{U}(x,y,z,t) = U_x(x,y,x,t)\,\widehat{i} + U_y(x,y,z,t)\,\widehat{j} + U_z(x,y,z,t)\,\widehat{k} \tag{8.32}$$

Therefor the acceleration will be

$$\frac{D\boldsymbol{U}}{Dt} = \frac{d\,U_x}{dt}\,\widehat{i} + \frac{d\,U_y}{dt}\,\widehat{j} + \frac{d\,U_z}{dt}\,\widehat{k} \tag{8.33}$$

The velocity components are a function of four variables, ($x$, $y$, $z$, and $t$), and hence

$$\frac{D\,U_x}{Dt} = \frac{\partial U_x}{\partial t}\overbrace{\frac{d\,t}{d\,t}}^{=1} + \frac{\partial U_x}{\partial x}\overbrace{\frac{d\,x}{d\,t}}^{U_x} + \frac{\partial U_x}{\partial y}\overbrace{\frac{d\,y}{d\,t}}^{U_y} + \frac{\partial U_x}{\partial z}\overbrace{\frac{d\,z}{d\,t}}^{U_z} \tag{8.34}$$

The acceleration in the $x$ can be written as

$$\frac{D\,U_x}{Dt} = \frac{\partial U_x}{\partial t} + U_x\frac{\partial U_x}{\partial x} + U_y\frac{\partial U_x}{\partial y} + U_z\frac{\partial U_x}{\partial z} = \frac{\partial U_x}{\partial t} + (\boldsymbol{U}\cdot\nabla)\,U_x \tag{8.35}$$

The same can be developed to the other two coordinates which can be combined (in a vector form) as

$$\frac{d\boldsymbol{U}}{dt} = \frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} \tag{8.36}$$

or in a more explicit form as

$$\frac{d\boldsymbol{U}}{dt} = \overbrace{\frac{\partial \boldsymbol{U}}{\partial t}}^{\substack{\text{local}\\ \text{acceleration}}} + \overbrace{\boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial x} + \boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial y} + \boldsymbol{U}\frac{\partial \boldsymbol{U}}{\partial z}}^{\substack{\text{convective}\\ \text{acceleration}}} \tag{8.37}$$

The time derivative referred in the literature as the local acceleration which vanishes when the flow is in a steady state. While the flow is in a steady state there is only convecive acceleration of the flow. The flow in a nozzle is an example to flow at steady state but yet has acceleration which flow with a very low velocity can achieve a supersonic flow.

## 8.4 Momentum Conservation

The relationship among the shear stress various components have to be established. The stress is a relationship between the force and area it is acting on or force divided by the area (division of vector by a vector). This division creates a tensor which the physical meaning will be explained here (the mathematical explanation can be found in the mathematical appendix of the book). The area has a direction or orientation which control the results of this division. So it can be written that

$$\boldsymbol{\tau} = f(\boldsymbol{F}, \boldsymbol{A}) \tag{8.38}$$

It was shown that in a static case (or in better words, when the shear stresses are absent) it was written

$$\boldsymbol{\tau} = -P\widehat{n} \tag{8.39}$$

It also was shown that the pressure has to be continuous. However, these stresses that act on every point and have three components on every surface and depend on the surface orientation. A common approach is to collect the stress in a "standard" orientation and then if needed the stresses can be reorientated to a new direction. The transformation is available because the "standard" surface can be transformed using trigonometrical functions. In Cartesian coordinates on surface in the $x$ direction the stresses are

$$\boldsymbol{\tau}^{(x)} = \quad \tau_{xx} \quad \tau_{xy} \quad \tau_{xz} \tag{8.40}$$

where $\tau_{xx}$ is the stress acting on surface $x$ in the $x$ direction, and $\tau_{xy}$ is the stress acting on surface $x$ in the $y$ direction, similarly for $\tau_{xz}$. The notation $\boldsymbol{\tau}^{(x_i)}$ is used to denote the stresses on $x_i$ surface. It can be noticed that no mathematical symbols are written between the components. The reason for this omission is that there is no physical meaning for it[9]. Similar "vectors" exist for the $y$ and $z$ coordinates which can

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[9] It can be argue that there is physical meaning that does not significant to the understanding of the subject.

be written in a matrix form

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{8.41}$$

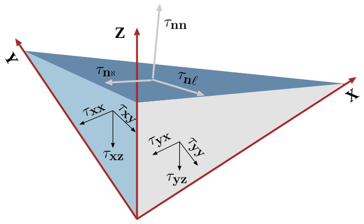

*Fig. -8.5. Stress diagram on a tetrahedron shape.*

Suppose that a straight angle tetrahedron is under stress as shown in Figure 8.5. The forces balance in the $x$ direction excluding the slanted surface is

$$F_x = -\tau_{yx}\delta A_y - \tau_{xx}\delta A_x - \tau_{zx}\delta A_z \tag{8.42}$$

where $\delta A_y$ is the surface area of the tetrahedron in the $y$ direction, $\delta A_x$ is the surface area of the tetrahedron in the $x$ direction and $\delta A_z$ is the surface area of the tetrahedron in the $z$ direction. The opposing forces which acting on the slanted surface in the $x$ direction are

$$F_x = \delta A_n \left(\tau_{nn}\,\widehat{n}\cdot\widehat{i} - \tau_{n\ell}\,\widehat{\ell}\cdot\widehat{i} - \tau_{n\aleph}\widehat{\aleph}\cdot\widehat{i}\right) \tag{8.43}$$

Where here $\widehat{\aleph}$, $\widehat{\ell}$ and $\widehat{n}$ are the local unit coordinates on $n$ surface the same can be written in the $x$, and $z$ directions. The transformation matrix is then

$$\begin{pmatrix} F_x \\ F_y \\ F_x \end{pmatrix} = \begin{pmatrix} \widehat{n}\cdot\widehat{i} & \widehat{\ell}\cdot\widehat{i} & \widehat{\aleph}\cdot\widehat{i} \\ \widehat{n}\cdot\widehat{j} & \widehat{\ell}\cdot\widehat{j} & \widehat{\aleph}\cdot\widehat{j} \\ \widehat{n}\cdot\widehat{k} & \widehat{\ell}\cdot\widehat{k} & \widehat{\aleph}\cdot\widehat{k} \end{pmatrix} \delta A_n \tag{8.44}$$

When the tetrahedron is shrunk to a point relationship of the stress on the two sides can be expended by Taylor series and keeping the first derivative. If the first derivative is neglected (tetrahedron is without acceleration) the two sides are related as

$$-\tau_{yx}\delta A_y - \tau_{xx}\delta A_x - \tau_{zx}\delta A_z = \delta A_n \left(\tau_{nn}\,\widehat{n}\cdot\widehat{i} - \tau_{n\ell}\,\widehat{\ell}\cdot\widehat{i} - \tau_{n\aleph}\widehat{\aleph}\cdot\widehat{i}\right) \tag{8.45}$$

The same can be done for $y$ and $z$ directions. The areas are related to each other through angles. These relationships provide the transformation for the different orientations which depends only angles of the orientations. This matrix is referred to as stress tensor and as it can be observed has nine terms.

### The Symmetry of the Stress Tensor

A small liquid cubical has three possible rotation axes. Here only one will be discussed the same conclustions can be drown on the other direction. The cubical rotation can involve two parts: one distortion and one rotation[10]. A finite angular distortion of

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[10] For infinitesimal change the lines can be approximated as straight.

infinitesimal cube requires an infinite shear which required for infinite moment. Hence, the rotation of the infinitesimal fluid cube can be viewed as it is done almost as a solid body rotation. Balance of momentum around the $z$ direction shown in Figure 8.6 is

$$M_z = I_{zz} \frac{d\theta}{dt} \tag{8.46}$$

Where $M_z$ is the cubic moment around the cubic center and $I_{zz}$[11] is the moment of inertia around that center. The momentum can be assested by the shear stresses which act on it. The shear stress at point $x$ is $\tau_{xy}$. However, the shear stress at point $x + dx$ is

$$\tau_{xy}\big|_{x+dx} = \tau_{xy} + \frac{d\tau_{xy}}{dx} dx \tag{8.47}$$

The same can be said for $\tau_{yx}$ for $y$ direction. The clarity of this analysis can be improved if additional terms are taken, yet it turn out that the results will be the same. The normal body force (gravity) acts through the cubic center of gravity. The moment that creats by this action can be neglected (the changes are insignificant). However, for cases that body force, such as the magnetic fields, can create torque. For simplicity and generality, it is assumed that the external body force exerts a torque $G_T$ per unit volume at the specific location. The body force can exert torque is due to the fact that the body force is not uniform and hence not act through the mass center.

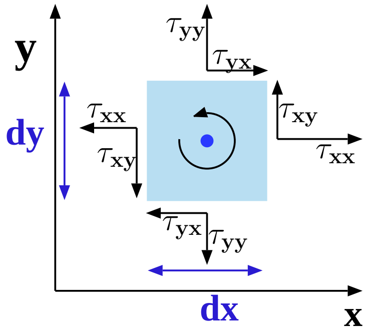

*Fig. -8.6. Diagram to analysis the shear stress tensor.*

*Advance material can be skipped*

The shear stress in the surface direction potentially can result in the torque due to the change in the shear stress[12]. For example, $\tau_{xx}$ at $x$ can be expended as a linear function

$$\tau_{xx} = \tau_{xx}\big|_y + \left.\frac{d\tau_{xx}}{dy}\right|_y \eta \tag{8.48}$$

where $\eta$ is the local coordinate in the $y$ direction stating at $y$ and "mostly used" between $y < \eta < y + dy$.

---

[11] See for the derivations in Example 3.5 for moment of inertia.

[12] This point bother this author in the completeness of the proof. It can be ignored, but provided to those who wonder why body forces can contribute to the torque while pressure, even though varied, does not. This point is for self convincing since it deals with a "strange" and problematic "animals" of integral of infinitesimal length.

The moment that results from this shear force (clockwise positive) is

$$\int_{y}^{y+dy} \tau_{xx}(\eta) \left(\eta - \frac{dy}{2}\right) d\eta \tag{8.49}$$

Substituting (8.48) into (8.49) results

$$\int_{y}^{y+dy} \left( \tau_{xx}\big|_{y} + \left.\frac{d\tau_{xx}}{dy}\right|_{y} \eta \right) \left(\eta - \frac{dy}{2}\right) d\eta \tag{8.50}$$

The integral of (8.50) isn't zero (non symmetrical function around the center of integration). The reason that this term is neglected because on the other face of the cubic contributes an identical term but in the opposing direction (see Figure 8.6).

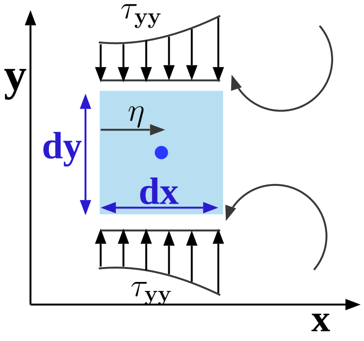

*Fig. -8.7. The shear stress creating torque.*

*End Advance material*

The net torque in the z-direction around the particle's center would then be

$$\begin{gathered}(\tau_{yx}) \frac{dx\,dy\,dz}{2} - \left(\tau_{yx} + \frac{\partial \tau_{xy}}{\partial x}\right) \frac{dx\,dy\,dz}{2} + (\tau_{xy}) \frac{dx\,dy\,dz}{2} - \\ \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\right) \frac{dx\,dy\,dz}{2} = \overbrace{\rho\, dx\, dy\, dz \left((dx)^2 + (dy)^2\right)}^{I_{zz}} \frac{d\theta}{dt}\end{gathered} \tag{8.51}$$

The actual components which contribute to the moment are

$$G_T + \tau_{xy} - \tau_{xy} + \frac{\overbrace{\partial \left(\tau_{yx} - \tau_{xy}\right)}^{\cong 0}}{\partial y} = \rho \frac{\underbrace{\left((dx)^2 + (dy)^2\right)}_{=0}}{12} \frac{d\theta}{dt} \tag{8.52}$$

which means since that $dx \longrightarrow 0$ and $dy \longrightarrow 0$ that

$$G_T + \tau_{xy} = \tau_{yx} \tag{8.53}$$

This analysis can be done on the other two directions and hence the general conclusion is that

$$G_T + \tau_{ij} = \tau_{ji} \tag{8.54}$$

where $i$ is one of $x, y, z$ and the $j$ is any of the other $x, y, z$[13]. For the case of $G_T = 0$ the stress tensor becomes symmetrical. The gravity is a body force that is considered in many kind of calculations and this force cause a change in symmetry of the stress

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[13] The index notation is not the main mode of presentation in this book. However, since Potto Project books are used extensively and numerous people asked to include this notation it was added. It is believed that this notation should and can be used only after the physical meaning was "digested."

tensor. However, this change, for almost all practical purposes, can be neglected[14]. The magnetic body forces on the other hand are significant and have to be included in the calculations. If the body forces effect is neglected or do not exist in the problem then regardless the coordinate system orientation

$$\tau_{ij} = \tau_{ji} \qquad (i \neq j) \tag{8.55}$$

## 8.5 Derivations of the Momentum Equation

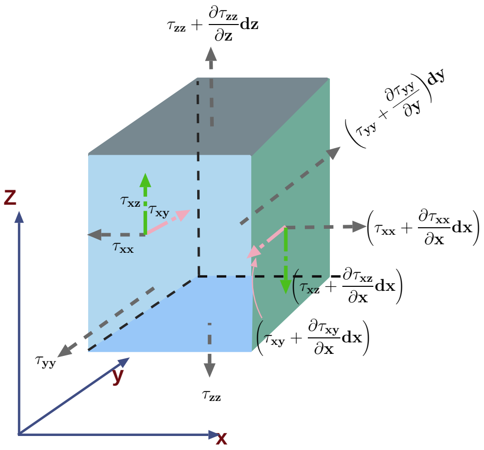


*Fig. -8.8. The shear stress at different surfaces. All shear stress shown in surface $x$ and $x+dx$.*

Previously it was shown that equation (6.11) is equivalent to Newton second law for fluids. Equation (6.11) is also applicable for the small infinitesimal cubic. One direction of the vector equation will be derived for $x$ Cartesian coordinate (see Figure 8.8). Later Newton second law will be used and generalized. For surface forces that acting on the cubic are surface forces, gravitation forces (body forces), and internal forces. The body force that acting on infinitesimal cubic in $x$ direction is

$$\widehat{i} \cdot \boldsymbol{f}_B = \boldsymbol{f}_{B_x}\, dx\, dy\, dz \tag{8.56}$$

The dot product yields a force in the directing of $x$. The surface forces in $x$ direction on the $x$ surface on are

$$f_{xx} = \left.\tau_{xx}\right|_{x+dx} \times \overbrace{dy\, dz}^{dA_x} - \left.\tau_{xx}\right|_{x} \times \overbrace{dy\, dz}^{dA_x} \tag{8.57}$$

---

[14] In the Dimensional Analysis a discussion about this effect hopefully will be presented.

The surface forces in $x$ direction on the $y$ surface on are

$$f_{xy} = \tau_{yx}\big|_{y+dy} \times \overbrace{dx\,dz}^{dA_y} - \tau_{yx}\big|_{y} \times \overbrace{dx\,dz}^{dA_y} \tag{8.58}$$

The same can be written for the $z$ direction. The shear stresses can be expanded into Taylor series as

$$\tau_{ix}\big|_{i+di} = \tau_{ix} + \left.\frac{\partial\left(\tau_{ix}\right)}{\partial i}\right|_{i} di + \cdots \tag{8.59}$$

where $i$ in this case is $x$, $y$, or $z$. Hence, the total net surface force results from the shear stress in the $x$ direction is

$$f_x = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dx\,dy\,dz \tag{8.60}$$

after rearrangement equations such as (8.57) and (8.58) transformed into

$$\overbrace{\frac{DU_x}{Dt}\,\rho\,\cancel{dx}\,\cancel{dy}\,\cancel{dz}}^{\text{internal forces}} = \overbrace{\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\cancel{dx}\,\cancel{dy}\,\cancel{dz}}^{\text{surface forces}} + \overbrace{f_{G\,x}\,\rho\,\cancel{dx}\,\cancel{dy}\,\cancel{dz}}^{\text{body forces}} \tag{8.61}$$

equivalant equation (8.61) for $y$ coordinate is

$$\rho\,\frac{DU_y}{Dt} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho\,f_{G\,y} \tag{8.62}$$

The same can be obtained for the $z$ component

$$\rho\,\frac{DU_z}{Dt} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho\,f_{G\,z} \tag{8.63}$$

*Advance material can be skipped*

Generally the component momentum equation is as

$$\rho\,\frac{DU_i}{Dt} = \left(\frac{\partial \tau_{ii}}{\partial i} + \frac{\partial \tau_{ji}}{\partial j} + \frac{\partial \tau_{ki}}{\partial j}\right) + \rho\,f_{G\,i} \tag{8.64}$$

*End Advance material*

Where $i$ is the balance direction and $j$ and $k$ are two other coordinates. Equation (8.64) can be written in a vector form which combined all three components into one equation. The advantage of the vector from allows the usage of the different coordinates. The vector form is

$$\rho\,\frac{D\boldsymbol{U}}{Dt} = \nabla\cdot\boldsymbol{\tau}^{(i)} + \rho\,\boldsymbol{f_G} \tag{8.65}$$

where here

$$\boldsymbol{\tau}^{(i)} = \tau_{ix}\widehat{i} + \tau_{iy}\widehat{j} + \tau_{iz}\widehat{k}$$

is part of the shear stress tensor and $i$ can be any of the $x, y$, or $z$.

Or in index (Einstein) notation as

$$\rho \frac{DU_i}{Dt} = \frac{\partial \tau_{ji}}{\partial x_i} + \rho\, f_{G\,i} \tag{8.66}$$

*End Advance material*

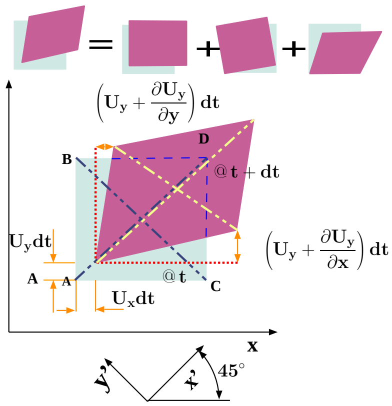


*Fig. -8.9. Control volume at $t$ and $t + dt$ under continuous angle deformation. Notice the three combinations of the deformation shown by purple color relative to blue color.*

Equations (8.61) or (8.62) or (8.63) requires that the stress tensor be defined in term of the velocity/deformaiton. The relationship between the stress tensor and deformation depends on the classes of materials the stresses acts on. Additionally, the deformation can be viewed as a function of the velocity field. As engineers do in general, the simplest model is assumed which referred as the solid continuum model. In this model the relationship bewtween the (shear) stresses and rate of strains are assumed to be linear. In solid material, the shear stress yields a fix amount of deformation. In contrast, when applying the shear stress in fluids, the result is a continuous deformation. Furthermore, reduction of the shear stress does not return the material to its original state as in solids. The similarity to solids the increase shear stress in fluids yields larger deformations. Thus this "solid" model is a linear relationship with three main assumptions:

a. There is no preference in the orientation (also call isentropic fluid),

b. there is no left over stresses (In other words when the "no shear stress" situation exist the rate of deformation or strain is zero), and

c. a linear relationship exist between the shear stress and the rate of shear strain.

At time $t$, the control volume is at a square shape and at a location as depicted in Figure 8.9 (by the blue color). At time $t + dt$ the control volume undergoes three different changes. The control volume moves to a new location, rotates and changes the shape (the purpule color in in Figure 8.9). The translational movement is referred to a movement of body without change of the body and without rotation. The rotation is the second movement that referred to a change in of the relative orientation inside the control volume. The third change is the misconfiguration or control volume (deformation). The deformation of the control volume has several components (see the top of Figure 8.9). The shear stress is related to the change in angle of the control volume lower left corner. The angle between $x$ to the new location of the control volume can be approximate for a small angle as

$$\frac{d\gamma_x}{dt} = \tan\left(\frac{U_y + \frac{dU_y}{dx}dx - U_y}{dx}\right) = \tan\left(\frac{dU_y}{dx}\right) \cong \frac{dU_y}{dx} \tag{8.67}$$

The total angle deformation (two sides $x$ and $y$) is

$$\frac{D\gamma_{xy}}{Dt} = \frac{dU_y}{dx} + \frac{dU_x}{dy} \tag{8.68}$$

In these derivatives, the symmetry $\frac{dU_y}{dx} \neq \frac{dU_x}{dy}$ was not assumed and or required because rotation of the control volume. However, under isentropic material it is assumed that all the shear stresses contribute equally. For the assumption of a linear fluid[15].

$$\tau_{xy} = \mu\frac{D\gamma_{xy}}{Dt} = \mu\left(\frac{dU_y}{dx} + \frac{dU_x}{dy}\right) \tag{8.69}$$

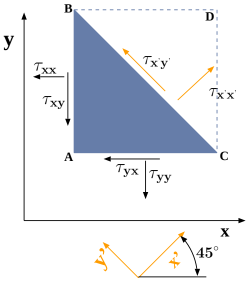


*Fig. -8.10. Shear stress at two coordinates in $45^\circ$ orientations.*

where, $\mu$ is the "normal" or "ordinary" viscosity coefficient which relates the linear coefficient of proportionality and shear stress. This deformation angle coefficient is assumed to be a property of the fluid. In a similar fashion it can be written to other directions for $x\,z$ as

$$\tau_{xz} = \mu\frac{D\gamma_{xz}}{Dt} = \mu\left(\frac{dU_z}{dx} + \frac{dU_x}{dz}\right) \tag{8.70}$$

and for the directions of $y\,z$ as

$$\tau_{yz} = \mu\frac{D\gamma_{yz}}{Dt} = \mu\left(\frac{dU_z}{dy} + \frac{dU_y}{dz}\right) \tag{8.71}$$

Note that the viscosity coefficient (the linear coefficient[16]) is assumed to be the same regardless of the direction. This assumption is referred as isotropic viscosity. It can be noticed at this stage, the relationship for the two of stress tensor parts was established. The only missing thing, at this stage, is the diagonal component which to be dealt below.

— — *Advance material can be skipped* — —

In general equation (8.69) can be written as

$$\tau_{ij} = \mu\frac{D\gamma_{ij}}{Dt} = \mu\left(\frac{dU_j}{di} + \frac{dU_i}{dj}\right) \tag{8.72}$$

---

[15] While not marked as important equation this equation is the source of the derivation.

[16] The first assumption was mentioned above.

where $i \neq j$ and $i = x$ or $y$ or $z$.

*End Advance material*

### Normal Stress

The normal stress, $\tau_{ii}$ (where $i$ is either ,$x$, $y$, $z$) appears in the shear matrix diagonal. To find the main (or the diagonal) stress the coordinates are rotate by $45°$. The diagonal lines (line *BC* and line *AD* in Figure 8.9) in the control volume move to the new locations. In addition, the sides *AB* and *AC* rotate in unequal amount which make one diagonal line longer and one diagonal line shorter. The normal shear stress relates to the change in the diagonal line length change. This relationship can be obtained by changing the coordinates orientation as depicted by Figure 8.10. The $dx$ is constructed so it equals to $dy$. The forces acting in the direction of $x'$ on the ellement are combination of several terms. For example, on the "$x$" surface (lower surface) and the "$y$" (left) surface, the shear stresses are acting in this direction. It can be noticed that "$dx$" surface is $\sqrt{2}$ times larger than $dx$ and $dy$ surfaces. The force balance in the $x'$ is

$$\overbrace{dy}^{A_x}\,\tau_{xx}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_x} + \overbrace{dx}^{A_y}\,\tau_{yy}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} + \overbrace{dx}^{A_y}\,\tau_{yx}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} + \overbrace{dy}^{A_x}\,\tau_{xy}\,\overbrace{\frac{1}{\sqrt{2}}}^{\cos\theta_y} = \overbrace{dx\sqrt{2}}^{A_{x'}}\,\tau_{x'x'} \tag{8.73}$$

dividing by $dx$ and after some rearrangements utilizing the identity $\tau_{xy} = \tau_{yx}$ results in

$$\frac{\tau_{xx} + \tau_{yy}}{2} + \tau_{yx} = \tau_{x'x'} \tag{8.74}$$

Setting the similar analysis in the $y'$ results in

$$\frac{\tau_{xx} + \tau_{yy}}{2} - \tau_{yx} = \tau_{y'y'} \tag{8.75}$$

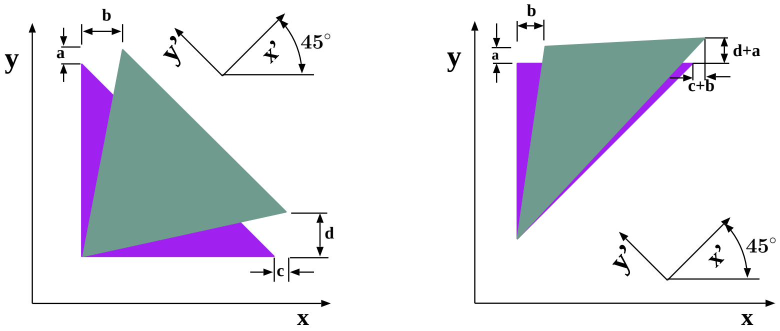
Subtracting (8.75) from (8.74) results in

$$2\,\tau_{yx} = \tau_{x'x'} - \tau_{y'y'} \tag{8.76}$$

or dividing by $2$ equation (8.76) becomes

$$\tau_{yx} = \frac{1}{2}\left(\tau_{x'x'} - \tau_{y'y'}\right) \tag{8.77}$$

Equation (8.76) relates the difference between the normal shear stress and the normal shear stresses in $x'$, $y'$ coordinates) and the angular strain rate in the regular $(x, y$ coordinates). The linear deformations in the $x'$ and $y'$ directions which is rotated $45°$ relative to the $x$ and $y$ axes can be expressed in both coordinates system. The angular strain rate in the $(x,\ y)$ is frame related to the strain rates in the $(x',\ y')$ frame. Figure 8.11(a) depicts the deformations of the triangular particles between time $t$ and $t + dt$.



(a) Deformations of the isosceles triangular.

(b) Deformation of the straight angle triangle.

*Fig. -8.11. Different triangles deformation for the calculations of the normal stress.*

The small deformations a , b, c, and d in the Figure are related to the incremental linear strains. The rate of strain in the $x$ direction is

$$d\epsilon_x = \frac{c}{dx} \tag{8.78}$$

The rate of the strain in $y$ direction is

$$d\epsilon_y = \frac{a}{dx} \tag{8.79}$$

The total change in the deformation angle is related to $\tan\theta$, in both sides $(d/dx+b/dy)$ which in turn is related to combination of the two sides angles. The linear angular deformation in $xy$ direction is

$$d\gamma_{xy} = \frac{b+d}{dx} \tag{8.80}$$

Here, $d\epsilon_x$ is the linear strain (increase in length divided by length) of the particle in the $x$ direction, and $d\epsilon_y$ is its linear strain in the y-direction. The linear strain in the $x'$ direction can be computed by observing Figure 8.11(b). The hypotenuse of the triangle is oriented in the $x'$ direction (again observe Figure 8.11(b)). The original length of the hypotenuse $\sqrt{2}dx$. The change in the hypotenuse length is $\sqrt{(c+b)^2+(a+d)^2}$. It can be approximated that the change is about $45^\circ$ because changes are infinitesimally small. Thus, $\cos 45^\circ$ or $\sin 45^\circ$ times the change contribute as first approximation to change. Hence, the ratio strain in the $x'$ direction is

$$d\epsilon_{x'} = \frac{\sqrt{(c+b)^2+(a+d)^2}}{\sqrt{2}dx} \simeq \frac{\dfrac{(c+b)}{\sqrt{2}} + \dfrac{(c+b)}{\sqrt{2}} + \overbrace{f\left(dx'\right)}^{\sim 0}}{\sqrt{2}dx} \tag{8.81}$$

Equation (8.81) can be interpreted as (using equations (8.78), (8.79), and (8.80)) as

$$d\epsilon_{x'} = \frac{1}{2}\left(\frac{a+b+c+d}{dx}\right) = \frac{1}{2}\left(d\epsilon_y + d\epsilon_y + d\gamma_{xy}\right) \tag{8.82}$$

In the same fashion, the strain in $y'$ coordinate can be interpreted to be

$$d\epsilon_{y'} = \frac{1}{2}\left(d\epsilon_y + d\epsilon_y - d\gamma_{xy}\right) \tag{8.83}$$

Notice the negative sign before $d\gamma_{xy}$. Combining equation (8.82) with equation (8.83) results in

$$d\epsilon_{x'} - d\epsilon_{y'} = d\gamma_{xy} \tag{8.84}$$

Equation (8.84) describing in Lagrangian coordinates a single particle. Changing it to the Eulerian coordinates transforms equation (8.84) into

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{D\gamma_{xy}}{Dt} \tag{8.85}$$

From (8.69) it can be observed that the right hand side of equation (8.85) can be replaced by $\tau_{xy}/\mu$ to read

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{\tau_{xy}}{\mu} \tag{8.86}$$

From equation (8.76) $\tau_{xy}$ be substituted and equation (8.86) can be continued and replaced as

$$\frac{D\epsilon_{x'}}{Dt} - \frac{D\epsilon_{y'}}{Dt} = \frac{1}{2\,\mu}\left(\tau_{x'x'} - \tau_{y'y'}\right) \tag{8.87}$$

Figure 8.12 depicts the approximate linear deformation of the element. The linear deformation is the difference between the two sides as

$$\frac{D\epsilon_{x'}}{Dt} = \frac{\partial U_{x'}}{\partial x'} \tag{8.88}$$

The same way it can written for the $y'$ coordinate.

$$\frac{D\epsilon_{y'}}{Dt} = \frac{\partial U_{y'}}{\partial y'} \tag{8.89}$$

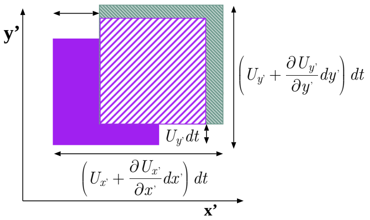


*Fig. -8.12. Linear strain of the element purple denotes $t$ and blue is for $t + dt$. Dashed squares denotes the movement without the linear change.*

Equation (8.88) can be written in the $y'$ and is similar by substituting the coordinates. The rate of strain relations can be substituted by the velocity and equations (8.88) and (8.89) changes into

$$\tau_{x'x'} - \tau_{y'y'} = 2\mu \left( \frac{\partial U_{x'}}{\partial x'} - \frac{\partial U_{y'}}{\partial y'} \right) \tag{8.90}$$

Similar two equations can be obtained in the other two plans. For example in $y'$–$z'$ plan one can obtained

$$\tau_{x'x'} - \tau_{z'z'} = 2\mu \left( \frac{\partial U_{x'}}{\partial x'} - \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.91}$$

Adding equations (8.90) and (8.91) results in

$$\overbrace{(3-1)}^{2} \tau_{x'x'} - \tau_{y'y'} - \tau_{z'z'} = \overbrace{(6-2)}^{4} \mu \frac{\partial U_{x'}}{\partial x'} - 2\,\mu \left( \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.92}$$

rearranging equation (8.92) transforms it into

$$3\,\tau_{x'x'} = \tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'} + 6\,\mu \frac{\partial U_{x'}}{\partial x'} - 2\,\mu \left( \frac{\partial U_{x'}}{\partial x'} + \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.93}$$

Dividing the restuls by 3 so that one can obtained the following

$$\tau_{x'x'} = \overbrace{\frac{\tau_{x'x'} + \tau_{y'y'} + \tau_{z'z'}}{3}}^{\text{“mechanical” pressure}} + 2\,\mu \frac{\partial U_{x'}}{\partial x'} - \frac{2}{3}\,\mu \left( \frac{\partial U_{x'}}{\partial x'} + \frac{\partial U_{y'}}{\partial y'} + \frac{\partial U_{z'}}{\partial z'} \right) \tag{8.94}$$

The “mechanical” pressure, $P_m$, is defined as the (negative) average value of pressure in directions of $x'$–$y'$–$z'$. This pressure is a true scalar value of the flow field since the propriety is averaged or almost[17] invariant to the coordinate transformation. In situations where the main diagonal terms of the stress tensor are not the same in all directions (in some viscous flows) this property can be served as a measure of the local normal stress. The mechanical pressure can be defined as averaging of the normal stress acting on a infinitesimal sphere. It can be shown that this two definitions are “identical” in the limits[18]. With this definition and noticing that the coordinate system $x'$–$y'$ has no special significance and hence equation (8.94) must be valid in any coordinate system thus equation (8.94) can be written as

$$\tau_{xx} = -P_m + 2\,\mu \frac{\partial U_x}{\partial x} + \frac{2}{3} \mu \nabla \cdot \boldsymbol{U} \tag{8.95}$$

Again where $P_m$ is the mechanical pressure and is defined as

Mechanical Pressure

$$P_m = -\frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3} \tag{8.96}$$

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[17] It identical only in the limits to the mechanical measurements.

[18] G. K. Batchelor, An Introduction to Fluid Mechanics, Cambridge University Press, 1967, p.141.

It can be observed that the non main (diagonal) terms of the stress tensor are represented by an equation like (8.72). Commonality engineers like to combined the two difference expressions into one as

$$\tau_{xy} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U}\right)\overbrace{\delta_{xy}}^{=0} + \mu\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) \tag{8.97}$$

or

$$\tau_{xx} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U}\right)\overbrace{\delta_{xy}}^{=1} + \mu\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y}\right) \tag{8.98}$$

*Advance material can be skipped*

or index notation

$$\tau_{ij} = -\left(P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U}\right)\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{8.99}$$

*End Advance material*

where $\delta_{ij}$ is the Kronecker delta what is $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. While this expression has the advantage of compact writing, it does not add any additional information. This expression suggests a new definition of the thermodynamical pressure is

Thermodynamic Pressure

$$P = P_m + \frac{2}{3}\mu\nabla\cdot\boldsymbol{U} \tag{8.100}$$

### Summary of The Stress Tensor

The above derivations were provided as a long mathematical explanation[19]. To reduced one unknown (the shear stress) equation (8.61) the relationship between the stress tensor and the velocity were to be established. First, connection between $\tau_{xy}$ and the deformation was built. Then the association between normal stress and perpendicular stress was constructed. Using the coordinates transformation, this association was established. The linkage between the stress in the rotated coordinates to the deformation was established.

### Second Viscosity Coefficient

The coefficient $2/3\mu$ is experimental and relates to viscosity. However, if the derivations before were to include additional terms, an additional correction will be needed. This correction results in

$$P = P_m + \lambda\nabla\cdot\boldsymbol{U} \tag{8.101}$$

---

[19] Since the publishing the version 0.2.9.0 several people ask this author to summarize conceptually the issues. With God help, it will be provide before version 0.3.1

The value of $\lambda$ is obtained experimentally. This coefficient is referred in the literature by several terms such as the "expansion viscosity" "second coefficient of viscosity" and "bulk viscosity." Here the term bulk viscosity will be adapted. The dimension of the bulk viscosity, $\lambda$, is similar to the viscosity $\mu$.According to second law of thermodynamic derivations (not shown here and are under construction) demonstrate that $\lambda$ must be positive. The thermodynamic pressure always tends to follow the mechanical pressure during a change. The expansion rate of change and the fluid molecular structure through $\lambda$ control the difference. Equation (8.101) can be written in terms of the thermodynamic pressure $P$, as

$$\tau_{ij} = -\left[P + \left(\frac{2}{3}\mu - \lambda\right)\nabla\cdot\boldsymbol{U}\right]\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{8.102}$$

The significance of the difference between the thermodynamic pressure and the mechanical pressure associated with fluid dilation which connected by $\nabla\cdot\boldsymbol{U}$. The physical meaning of $\nabla\cdot\boldsymbol{U}$ represents the relative volume rate of change. For simple gas (dilute monatomic gases) it can be shown that $\lambda$ vanishes. In material such as water, $\lambda$ is large (3 times $\mu$) but the net effect is small because in that cases $\nabla\cdot\boldsymbol{U} \longrightarrow 0$. For complex liquids this coefficient, $\lambda$, can be over 100 times larger than $\mu$. Clearly for incompressible flow, this coefficient or the whole effect is vanished[20]. In most cases, the total effect of the dilation on the flow is very small. Only in micro fluids and small and molecular scale such as in shock waves this effect has some significance. In fact this effect is so insignificant that there is difficulty in to construct experiments so this effect can be measured. Thus, neglecting this effect results in

$$\tau_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{8.103}$$

To explain equation (8.103), it can be written for spesific coordinates. For example, for the $\tau_{xx}$ it can be written that

$$\tau_{xx} = -P + 2\frac{\partial U_x}{\partial x} \tag{8.104}$$

and the $y$ coordinate the equation is

$$\tau_{yy} = -P + 2\frac{\partial U_y}{\partial y} \tag{8.105}$$

however the mix stress, $\tau_{xy}$, is

$$\tau_{xy} = \tau_{yx} = \left(\frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}\right) \tag{8.106}$$

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[20] The reason that the effect vanish is because $\nabla\cdot\boldsymbol{U} = 0$.

For the total effect, substitute equation (8.102) into equation (8.61) which results in

$$\rho\left(\frac{DU_x}{Dt}\right) = -\frac{\partial\left(P+\left(\frac{2}{3}\mu-\lambda\right)\nabla\cdot\boldsymbol{U}\right)}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right) \quad +\boldsymbol{f}_{Bx} \tag{8.107}$$

or in a vector form as

N-S in stationary Coordinates

$$\boxed{\rho\,\frac{D\boldsymbol{U}}{Dt} = -\nabla P + \left(\frac{1}{3}\mu+\lambda\right)\nabla\left(\nabla\cdot\boldsymbol{U}\right) + \mu\,\nabla^2\boldsymbol{U} + \boldsymbol{f}_B} \tag{8.108}$$

For in index form as

$$\rho\,\frac{DU_i}{Dt} = -\frac{\partial}{\partial x_i}\left(P+\left(\frac{2}{3}\mu-\lambda\right)\nabla\cdot\boldsymbol{U}\right) + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\right) + \boldsymbol{f}_{Bi} \tag{8.109}$$

For incompressible flow the term $\nabla\cdot\boldsymbol{U}$ vanishes, thus equation (8.108) is reduced to

Momentum for Incompressible Flow

$$\boxed{\rho\,\frac{D\boldsymbol{U}}{Dt} = -\nabla P + \mu\,\nabla^2\boldsymbol{U} + \boldsymbol{f}_B} \tag{8.110}$$

or in the index notation it is written

$$\rho\,\frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu\,\frac{\partial^2\boldsymbol{U}}{\partial x_i\partial x_j} + \boldsymbol{f}_{Bi} \tag{8.111}$$

The momentum equation in Cartesian coordinate can be written explicitly for $x$ coordinate as

$$\begin{aligned}\rho\left(\frac{\partial U_x}{\partial t}+\right. & \left. U_x\frac{\partial U_x}{\partial x}+U_y\frac{\partial U_x}{\partial y}+U_z\frac{\partial U_x}{\partial z}\right) = \\ & -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)+\rho g_x\end{aligned} \tag{8.112}$$

Where $g_x$ is the the body force in the $x$ direction $(\widehat{i}\cdot\boldsymbol{g})$. In the $y$ coordinate the momentum equation is

$$\begin{aligned}\rho\left(\frac{\partial U_y}{\partial t}+\right. & \left. U_x\frac{\partial U_y}{\partial x}+U_y\frac{\partial U_y}{\partial y}+U_z\frac{\partial U_y}{\partial z}\right) = \\ & -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y\end{aligned} \tag{8.113}$$

in $z$ coordinate the momentum equation is

$$\begin{aligned}\rho\left(\frac{\partial U_z}{\partial t}+\quad U_x\frac{\partial U_z}{\partial x}+U_y\frac{\partial U_z}{\partial y}+U_z\frac{\partial U_z}{\partial z}\right)&=\\-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 U_z}{\partial x^2}+\frac{\partial^2 U_z}{\partial y^2}+\frac{\partial^2 U_z}{\partial z^2}\right)+\rho g_z\end{aligned}\tag{8.114}$$

## 8.6 Boundary Conditions and Driving Forces

### 8.6.1 Boundary Conditions Categories

The governing equations that were developed earlier requires some boundary conditions and initial conditions. These conditions described physical situations that are believed or should exist or approximated. These conditions can be categorized by the velocity, pressure, or in more general terms as the shear stress conditions (mostly at the interface). For this discussion, the shear tensor will be separated into two categories, pressure (at the interface direction) and shear stress (perpendicular to the area). A common velocity condition is that the liquid has the same value as the solid interface velocity. In the literature, this condition is referred as the "no slip" condition. The solid surface is rough thus the liquid participles (or molecules) are slowed to be at the solid surface velocity. This boundary condition was experimentally observed under many conditions yet it is not universal true. The slip condition (as oppose to "no slip" condition) exist in situations where the scale is very small and the velocity is relatively very small. The slip condition is dealing with a difference in the velocity between the solid (or other material) and the fluid media. The difference between the small scale and the large scale is that the slip can be neglected in the large scale while the slip cannot be neglected in the small scale. In another view, the difference in the velocities vanishes as the scale increases.

Another condition which affects whether the slip condition exist is how rapidly of the velocity change. The slip condition cannot be ignored in some regions, when the flow is with a strong velocity fluctuations. Mathematically the "no slip" condition is written as

$$\hat{\mathbf{t}}\cdot(\boldsymbol{U}_{fluid}-\boldsymbol{U}_{boundary})=0 \tag{8.115}$$

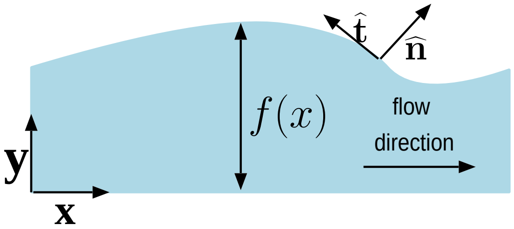

*Fig. -8.13. 1–Dimensional free surface describing $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{t}}$.*

where $\hat{\mathbf{n}}$ is referred to the area direction (perpendicular to the area see Figure 8.13). While this condition (8.115) is given in a vector form, it is more common to write this condition as a given velocity at a certain point such as

$$U(\ell)=U_\ell \tag{8.116}$$

Note, the "no slip" condition is applicable to the ideal fluid ("inviscid flows") because this kind of flow normally deals with large scales. The "slip" condition is written in similar fashion to equation (8.115) as

$$\widehat{\mathbf{t}} \cdot (\boldsymbol{U}_{fluid} - \boldsymbol{U}_{boundary}) = f(Q, scale, etc) \tag{8.117}$$

As oppose to a given velocity at particular point, a requirement on the acceleration (velocity) can be given in unknown position. The condition (8.115) can be mathematically represented in another way for free surface conditions. To make sure that all the material is accounted for in the control volume (does not cross the free surface), the relative perpendicular velocity at the interface must be zero. The location of the (free) moving boundary can be given as $f(\widehat{\boldsymbol{r}}, t) = 0$ as the equation which describes the bounding surface. The perpendicular relative velocity at the surface must be zero and therefore

$$\frac{Df}{Dt} = 0 \quad \text{on the surface } f(\widehat{\boldsymbol{r}}, t) = 0 \tag{8.118}$$

This condition is called the kinematic boundary condition. For example, the free surface in the two dimensional case is represented as $f(t, x, y)$. The condition becomes as

$$0 = \frac{\partial f}{\partial t} + U_x \frac{\partial f}{\partial x} + U_y \frac{\partial f}{\partial y} \tag{8.119}$$

The solution of this condition, sometime, is extremely hard to handle because the location isn't given but the derivative given on unknown location. In this book, this condition will not be discussed (at least not plane to be written).

The free surface is a special case of moving surfaces where the surface between two distinct fluids. In reality the interface between these two fluids is not a sharp transition but only approximation (see for the surface theory). There are situations where the transition should be analyzed as a continuous transition between two phases. In other cases, the transition is idealized an almost jump (a few molecules thickness). Furthermore, there are situations where the fluid (above one of the sides) should be considered as weightless material. In these cases the assumptions are that the transition occurs in a sharp line, and the density has a jump while the shear stress are continuous (in some cases continuously approach zero value). While a jump in density does not break any physical laws (at least those present in the solution), the jump in a shear stress (without a jump in density) does break a physical law. A jump in the shear stress creates infinite force on the adjoin thin layer. Off course, this condition cannot be tolerated since infinite velocity (acceleration) is impossible. The jump in shear stress can appear when the density has a jump in density. The jump in the density (between the two fluids) creates a surface tension which offset the jump in the shear stress. This condition is expressed mathematically by equating the shear stress difference to the forces results due to the surface tension. The shear stress difference is

$$\Delta\boldsymbol{\tau}^{(n)} = 0 = \Delta\boldsymbol{\tau}^{(n)}{}_{\substack{\text{upper}\\\text{surface}}} - \Delta\boldsymbol{\tau}^{(n)}{}_{\substack{\text{lower}\\\text{surface}}} \tag{8.120}$$

where the index $(n)$ indicate that shear stress are normal (in the surface area). If the surface is straight there is no jump in the shear stress. The condition with curved surface are out the scope of this book yet mathematically the condition is given as without explanation as

$$\widehat{\boldsymbol{n}} \cdot \boldsymbol{\tau}^{(n)} = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \tag{8.121}$$

$$\widehat{\boldsymbol{t}} \cdot \boldsymbol{\tau}^{(t)} = -\widehat{\boldsymbol{t}} \cdot \nabla\sigma \tag{8.122}$$

where $\widehat{\boldsymbol{n}}$ is the unit normal and $\widehat{\boldsymbol{t}}$ is a unit tangent to the surface (notice that direction pointed out of the "center" see Figure 8.13) and $R_1$ and $R_2$ are principal radii. One of results of the free surface condition (or in general, the moving surface condition) is that integration constant is unknown). In same instances, this constant is determined from the volume conservation. In index notation equation (8.121) is written[21] as

$$\tau_{ij}^{(1)}\, n_j + \sigma\, n_i \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = \tau_{ij}^{(2)}\, n_j \tag{8.123}$$

where $1$ is the upper surface and $2$ is the lower surface. For example in one dimensional[22]

$$\begin{aligned} \widehat{\boldsymbol{n}} = & \quad \frac{(-f'(x), 1)}{\sqrt{1 + (f'(x))^2}} \\ \widehat{\boldsymbol{t}} = & \quad \frac{(1, f'(x))}{\sqrt{1 + (f'(x))^2}} \end{aligned} \tag{8.124}$$

the unit vector is given as two vectors in $x$ and $y$ and the radius is given by equation (1.57). The equation is given by

$$\frac{\partial f}{\partial t} + U_x \frac{\partial f}{\partial x} = U_y \tag{8.125}$$

### The Pressure Condition

The second condition that commonality prescribed at the interface is the static pressure at a specific location. The static pressure is measured perpendicular to the flow direction. The last condition is similar to the pressure condition of prescribed shear stress or a relationship to it. In this category include the boundary conditions with issues of surface tension which were discussed earlier. It can be noticed that the boundary conditions that involve the surface tension are of the kind where the condition is given on boundary but no at a specific location.

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[21] There is no additional benefit in this writing, it just for completeness and can be ignored for most purposes.

[22] A one example of a reference not in particularly important or significant just a random example. Jean, M. Free surface of the steady flow of a Newtonian fluid in a finite channel. Arch. Rational Mech. Anal. 74 (1980), no. 3, 197–217.

### Gravity as Driving Force

The body forces, in general and gravity in a particular, are the condition that given on the flow beside the velocity, shear stress (including the surface tension) and the pressure. The gravity is a common body force which is considered in many fluid mechanics problems. The gravity can be considered as a constant force in most cases (see for dimensional analysis for the reasons).

### Shear Stress and Surface Tension as Driving Force

*Fig. -8.14. Kerosene lamp.*

If the fluid was solid material, pulling the side will pull all the material. In fluid (mostly liquid) shear stress pulling side (surface) will have limited effect and yet sometime is significant and more rarely dominate. Consider, for example, the case shown in Figure 8.14. The shear stress carry the material as if part of the material was a solid material. For example, in the kerosene lamp the burning occurs at the surface of the lamp top and the liquid is at the bottom. The liquid does not move up due the gravity (actually it is against the gravity) but because the surface tension.

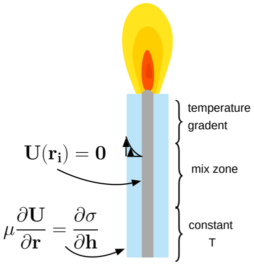

*Fig. -8.15. Schematic of kerosene lamp.*

The physical conditions in Figure 8.14 are used to idealize the flow around an inner rode to understand how to apply the surface tension to the boundary conditions. The fluid surrounds the rode and flows upwards. In that case, the velocity at the surface of the inner rode is zero. The velocity at the outer surface is unknown. The boundary condition at outer surface given by a jump of the shear stress. The outer diameter is depends on the surface tension (the larger surface tension the smaller the liquid diameter). The surface tension is a function of the temperature therefore the gradient in surface tension is result of temperature gradient. In this book, this effect is not discussed. However, somewhere downstream the temperature gradient is insignificant. Even in that case, the surface tension gradient remains. It can be noticed that, under the assumption presented here, there are two principal radii of the flow. One radius toward the center of the rode while the other radius is infinite (approximatly). In that case, the contribution due to the curvature is zero in the direction of the flow (see Figure 8.15). The only (almost) propelling source of the flow is the surface gradient ($\frac{\partial \sigma}{\partial n}$).

## 8.7 Examples for Differential Equation (Navier-Stokes)

Examples of an one-dimensional flow driven by the shear stress and pressure are presented. For further enhance the understanding some of the derivations are repeated. First, example dealing with one phase are present. Later, examples with two phase are presented.

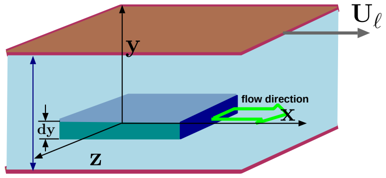

*Fig. -8.16. Flow between two plates, top plate is moving at speed of $U_\ell$ to the right (as positive). The control volume shown in darker colors.*

Example 8.6:
*Incompressible liquid flows between two infinite plates from the left to the right (as shown in Figure 8.16). The distance between the plates is $\ell$. The static pressure per length is given as $\Delta P$[23]. The upper surface is moving in velocity, $U_\ell$ (The rightside is defined as positive).*

Solution

In this example, the mass conservation yields

$$\overbrace{\frac{d}{dt}\int_{cv}\rho dV}^{=0} = -\int_{cv}\rho\, U_{rn} dA = 0 \tag{8.126}$$

The momentum is not accumulated (steady state and constant density). Further because no change of the momentum thus

$$\int_{A}\rho\, U_x\, U_{rn} dA = 0 \tag{8.127}$$

Thus, the flow in and the flow out are equal. It can concluded that the velocity in and out are the same (for constant density). The momentum conservation leads

$$-\int_{cv}\boldsymbol{P} dA + \int_{cv}\boldsymbol{\tau}_{xy} dA = 0 \tag{8.128}$$

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[23] The difference is measured at the bottom point of the plate.

The reaction of the shear stress on the lower surface of control volume based on Newtonian fluid is

$$\boldsymbol{\tau}_{xy} = -\mu \frac{dU}{dy} \tag{8.129}$$

On the upper surface is different by Taylor explanation as

$$\boldsymbol{\tau}_{xy} = \mu \left( \frac{dU}{dy} + \frac{d^2U}{dy^2}\,dy + \overbrace{\frac{d^3U}{dy^3}\,dy^2 + \cdots}^{\cong 0} \right) \tag{8.130}$$

The net effect of these two will be difference between them

$$\mu \left( \frac{dU}{dy} + \frac{d^2U}{dy^2}\,dy \right) - \mu \frac{dU}{dy} \cong \mu \frac{d^2U}{dy^2}\,dy \tag{8.131}$$

The assumptions is that there is no pressure difference in the $z$ direction. The only difference in the pressure is in the $x$ direction and thus

$$P - \left( P + \frac{dP}{dx}\,dx \right) = -\frac{dP}{dx}\,dx \tag{8.132}$$

A discussion why $\frac{\partial P}{\partial y} \sim 0$ will be presented later. The momentum equation in the $x$ direction (or from equation (8.112)) results (without gravity effects) in

$$-\frac{dP}{dx} = \mu \frac{d^2U}{dy^2} \tag{8.133}$$

Equation (8.133) was constructed under several assumptions which include the direction of the flow, Newtonian fluid. No assumption was imposed on the pressure distribution. Equation (8.133) is a partial differential equation but can be treated as ordinary differential equation in the $z$ direction of the pressure difference is uniform. In that case, the left hand side is equal to constant. The "standard" boundary conditions is non–vanishing pressure gradient (that is the pressure exist) and velocity of the upper or lower surface or both. It is common to assume that the "no slip" condition on the boundaries con-

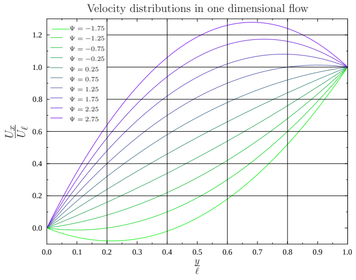

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*Fig. -8.17. One dimensional flow with a shear between two plates when $\Psi$ change value between -1.75 green line to 3 the blue line.*

dition[24]. The boundaries conditions are

$$\begin{aligned} U_x(y=0) &= 0 \\ U_x(y=\ell) &= U_\ell \end{aligned} \tag{8.134}$$

The solution of the "ordinary" differential equation (8.133) after the integration becomes

$$U_x = -\frac{1}{2}\frac{dP}{dx}\,y^2 + c_2\,y + c_3 \tag{8.135}$$

Applying the boundary conditions, equation (8.134)/ results in

$$U_x(y) = \frac{y}{\ell}\left(\overbrace{\frac{\ell^2}{U_0\,2\mu}\frac{dP}{dx}}^{=\Psi}\left(1-\frac{y}{\ell}\right)\right) + \frac{y}{\ell} \tag{8.136}$$

For the case where the pressure gradient is zero the velocity is linear as was discussed earlier in Chapter 1 (see Figure 8.17). However, if the plates or the boundary conditions do not move the solution is

$$U_x(y) = \left(\frac{\ell^2}{U_0\,2\mu}\frac{dP}{dx}\left(1-\frac{y}{\ell}\right)\right) + \frac{y}{\ell} \tag{8.137}$$

What happen when $\frac{\partial P}{\partial y} \sim 0$?

End Solution

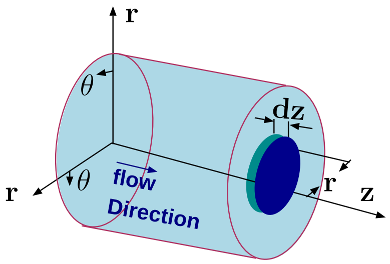

*Fig. -8.18. The control volume of liquid element in cylindrical coordinates.*

### Cylindrical Coordinates

Similarly the problem of one dimensional flow can be constructed for cylindrical coordinates. The problem is still one dimensional because the flow velocity is a function

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[24] A discussion about the boundary will be presented later.

of (only) radius. This flow referred as Poiseuille flow after Jean Louis Poiseuille a French Physician who investigated blood flow in veins. Thus, Poiseuille studied the flow in a small diameters (he was not familiar with the concept of Reynolds numbers). Rederivation are carried out for a short cut.

The momentum equation for the control volume depicted in the Figure 8.18a is

$$-\int \boldsymbol{P}\, dA + \int \boldsymbol{\tau} dA = \int \rho\, U_z\, U_{rn}\, dA \tag{8.138}$$

The shear stress in the front and back surfaces do no act in the $z$ direction. The shear stress on the circumferential part small dark blue shown in Figure 8.18a is

$$\int \boldsymbol{\tau}\, dA = \mu \frac{dU_z}{dr} \overbrace{2\,\pi\, r\, dz}^{dA} \tag{8.139}$$

The pressure integral is

$$\int \boldsymbol{P}\, dA = \left(P_{z_dz} - P_z\right) \pi\, r^2 = \left(P_z + \frac{\partial P}{\partial z} dz - P_z\right) \pi\, r^2 = \frac{\partial P}{\partial z} dz\, \pi\, r^2 \tag{8.140}$$

The last term is

$$\begin{aligned} &\int \rho\, U_z\, U_{rn}\, dA = \rho \int \quad U_z\, U_{rn}\, dA = \\ &\rho \left( \int_{z+dz} {U_{z+dz}}^2 dA \quad - \int_z {U_z}^2 dA \right) = \rho \int_z \left( {U_{z+dz}}^2 - {U_z}^2 \right) dA \end{aligned} \tag{8.141}$$

The term ${U_{z+dz}}^2 - {U_z}^2$ is zero because $U_{z+dz} = U_z$ because mass conservation conservation for any element. Hence, the last term is

$$\int \rho\, U_z\, U_{rn}\, dA = 0 \tag{8.142}$$

Substituting equation (8.139) and (8.140) into equation (8.138) results in

$$\mu \frac{dU_z}{dr} 2 \cancel{\pi} \cancel{r} \cancel{dz} = -\frac{\partial P}{\partial z} \cancel{dz} \cancel{\pi} r^{\cancel{2}} \tag{8.143}$$

Which shrinks to

$$\frac{2\,\mu}{r} \frac{dU_z}{dr} = -\frac{\partial P}{\partial z} \tag{8.144}$$

Equation (8.144) is a first order differential equation for which only one boundary condition is needed. The “no slip” condition is assumed

$$U_z(r = R) = 0 \tag{8.145}$$

Where $R$ is the outer radius of pipe or cylinder. Integrating equation (8.144) results in

$$U_z = -\frac{1}{\mu}\frac{\partial P}{\partial z}\, r^2 + c_1 \tag{8.146}$$

It can be noticed that asymmetrical element[25] was eliminated due to the smart short cut. The integration constant obtained via the application of the boundary condition which is

$$c_1 = -\frac{1}{\mu}\frac{\partial P}{\partial z}\, R^2 \tag{8.147}$$

The solution is

$$U_z = \frac{1}{\mu}\frac{\partial P}{\partial z}\, R^2 \left(1 - \left(\frac{r}{R}\right)^2\right) \tag{8.148}$$

While the above analysis provides a solution, it has several deficiencies which include the ability to incorporate different boundary conditions such as flow between concentering cyliders.

### Example 8.7:

*A liquid with a constant density is flowing between concentering cylinders as shown in Figure 8.19. Assume that the velocity at the surface of the cylinders is zero calculate the velocity profile. Build the velocity profile when the flow is one directional and viscosity is Newtonian. Calculate the flow rate for a given pressure gradient.*

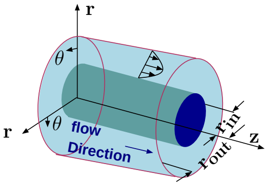

*Fig. -8.19. Liquid flow between concentric cylinders for example 8.7.*

SOLUTION

After the previous example, the appropriate version of the Navier–Stokes equation will be used. The situation is best suitable to solved in cylindrical coordinates. One of the solution of this problems is one dimensional solution. In fact there is no physical reason why the flow should be only one dimensional. However, it is possible to satisfy the boundary conditions. It turn out that the "simple" solution is the first mode that appear in reality. In this solution will be discussing the flow first mode. For this mode, the flow is assumed to be one dimensional. That is, the velocity isn't a function of the angle, or $z$ coordinate. Thus only equation in $z$ coordinate is needed. It can be noticed

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[25] Asymmetrical element or function is $-f(x) = f(-x)$

that this case is steady state and also the acceleration (convective acceleration) is zero

$$\rho\left(\overbrace{\frac{\partial U_z}{\partial t}}^{\neq f(t)} + \overbrace{U_r}^{=0}\frac{\partial U_z}{\partial r} + \overbrace{\frac{U_\phi}{r}}^{=0}\overbrace{\frac{\partial U_z}{\partial \phi}}^{U_z\neq f(\phi)} + U_z\overbrace{\frac{\partial U_z}{\partial z}}^{=0}\right) = 0 \tag{8.149}$$

The steady state governing equation then becames

$$\rho\left(\not{0}\right) = 0 = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_z}{\partial r}\right) + \overbrace{\cdots}^{=0}\right) + \cancel{\rho g_z} \tag{8.VII.a}$$

The PDE above (8.VII.a) required boundary conditions which are

$$\begin{aligned} U_z\left(r = r_i\right) = & \quad 0 \\ U_z\left(r = r_o\right) = & \quad 0 \end{aligned} \tag{8.VII.b}$$

Integrating equation (8.VII.a) once results in

$$r\frac{\partial U_z}{\partial r} = \frac{1}{2\,\mu}\frac{\partial P}{\partial z}r^2 + c_1 \tag{8.VII.c}$$

Dividing equation (8.VII.c) and integrating results for the second times results

$$\frac{\partial U_z}{\partial r} = \frac{1}{2\,\mu}\frac{\partial P}{\partial z}r + \frac{c_1}{r} \tag{8.VII.d}$$

Integration of equation (8.VII.d) results in

$$U_z = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}r^2 + c_1\,\ln r + c_2 \tag{8.VII.e}$$

Applying the first boundary condition results in

$$0 = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}{r_i}^2 + c_1\,\ln r_i + c_2 \tag{8.VII.f}$$

applying the second boundary condition yields

$$0 = \frac{1}{4\,\mu}\frac{\partial P}{\partial z}{r_o}^2 + c_1\,\ln r_o + c_2 \tag{8.VII.g}$$

The solution is

$$\begin{aligned} c_1 &= \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\frac{\partial P}{dz}\left({r_o}^2 - {r_i}^2\right) \\ c_2 &= \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\frac{\partial P}{dz}\left(\ln(r_i)\,{r_o}^2 - \ln(r_o)\,{r_i}^2\right) \end{aligned} \tag{8.VII.h}$$

The solution is when substituting the constats into equation (8.VII.e) results in

$$\begin{aligned} U_z(r) = \frac{1}{4\,\mu}\frac{\partial P}{\partial z} r^2 + \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left({r_o}^2 - {r_i}^2\right)\ln\,r \\ +\frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left(\ln(r_i)\,{r_o}^2 - \ln(r_o)\,{r_i}^2\right) \end{aligned} \tag{8.VII.i}$$

The flow rate is then

$$Q = \int_{r_i}^{r_o} U_z(r) dA \tag{8.VII.j}$$

Or substituting equation (8.VII.i) into equation (8.VII.j) transfomed into

$$\begin{aligned} Q = \int_A \left[\frac{1}{4\,\mu}\frac{\partial P}{\partial z} r^2 + \frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left({r_o}^2 - {r_i}^2\right)\ln\,r \right. \\ \left. +\frac{1}{4\,\mu}\,\ln\left(\frac{r_o}{r_i}\right)\,\frac{\partial P}{dz}\left(\ln(r_i)\,{r_o}^2 - \ln(r_o)\,{r_i}^2\right)\right] dA \end{aligned} \tag{8.VII.k}$$

A finite intergation of the last term in the integrand results in zero because it is constant. The integraion of the rest is

$$Q = \left[\frac{1}{4\,\mu}\frac{\partial P}{\partial z}\right] \int_{r_i}^{r_o} \left[r^2 + \ln\left(\frac{r_o}{r_i}\right)\left({r_o}^2 - {r_i}^2\right)\ln\,r\right] 2\,\pi\,r\,dr \tag{8.VII.l}$$

The first integration of the first part of the second squere bracket, $(r^3)$, is $1/4\left({r_o}^4 - {r_i}^4\right)$. The second part, of the second squere bracket, $(-a \times r \ln\,r)$ can be done by parts to be as

$$a\left(\frac{r^2}{4} - \frac{r^2\log\left(r\right)}{2}\right)$$

Applying all these "techniques" to equation (8.VII.l) results in

$$\begin{aligned} Q = \left[\frac{\pi}{2\,\mu}\frac{\partial P}{\partial z}\right]\left[\left(\frac{{r_o}^4}{4} - \frac{{r_i}^4}{4}\right) + \right. \\ \left. \ln\left(\frac{r_o}{r_i}\right)\quad\left({r_o}^2 - {r_i}^2\right)\left(\frac{{r_o}^2\ln\left(r_o\right)}{2} - \frac{{r_o}^2}{4} - \frac{{r_i}^2\ln\left(r_i\right)}{2} + \frac{{r_i}^2}{4}\right)\right] \end{aligned} \tag{8.VII.m}$$

The averaged velocity is obtained by dividing flow rate by the area $Q/A$.

$$U_{ave} = \frac{Q}{\pi\left({r_o}^2 - {r_i}^2\right)} \tag{8.150}$$

in which the identy of $(a^4 - b^4)/(a^2 - b^2)$ is $b^2 + a^2$ and hence

$$\begin{aligned} U_{ave} = \left[\frac{1}{2\,\mu}\frac{\partial P}{\partial z}\right]\left[\left(\frac{{r_o}^2}{4} + \frac{{r_i}^2}{4}\right) + \right. \\ \left. \ln\left(\frac{r_o}{r_i}\right)\quad\left(\frac{{r_o}^2\ln\left(r_o\right)}{2} - \frac{{r_o}^2}{4} - \frac{{r_i}^2\ln\left(r_i\right)}{2} + \frac{{r_i}^2}{4}\right)\right] \end{aligned} \tag{8.VII.n}$$

End Solution

The next example deals with the gravity as body force in two dimensional flow. This problem study by Nusselt[26] which developed the basics equations. This problem is related to many industrial process and is fundamental in understanding many industrial processes. Furthermore, this analysis is a building bloc for heat and mass transfer understanding[27].

### Example 8.8:

*In many situations in nature and many industrial processes liquid flows downstream on inclined plate at $\theta$ as shown in Figure 8.20. For this example, assume that the gas density is zero (located outside the liquid domain). Assume that "scale" is large enough so that the "no slip" condition prevail at the plate (bottom). For simplicity, assume that the flow is two dimensional. Assume that the flow obtains a steady state after some length (and the acceleration vanished). The dominate force is the gravity. Write the governing equations for this situation. Calculate the velocity profile. Assume that the flow is one dimensional in the $x$ direction.*

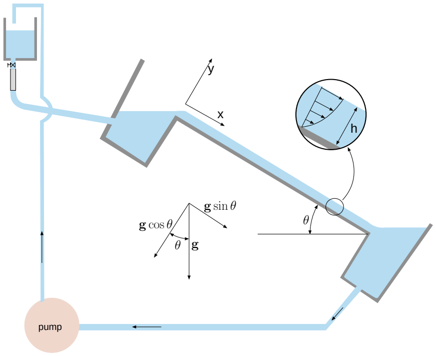

*Fig. -8.20. Mass flow due to temperature difference for example 8.1*

Solution

This problem is sutiable to Cartesian coordinates in which $x$ coordinate is pointed in the flow direction and $y$ perpendicular to flow direction (depicted in Figure 8.20). For this system, the gravity in the $x$ direction is $g\sin\theta$ while the direction of $y$ the gravity is $g\cos\theta$. The governing in the $x$ direction is

$$\rho\left(\overbrace{\frac{\partial U_x}{\partial t}}^{\neq f(t)} + U_x \overbrace{\frac{\partial U_x}{\partial x}}^{=0} + \overbrace{U_y}^{=0}\frac{\partial U_x}{\partial y} + \overbrace{U_z}^{-0}\frac{\partial U_x}{\partial z}\right) = - \overbrace{\frac{\partial P}{\partial x}}^{\sim 0} + \mu\left(\overbrace{\frac{\partial^2 U_x}{\partial x^2}}^{=0} + \frac{\partial^2 U_x}{\partial y^2} + \overbrace{\frac{\partial^2 U_x}{\partial z^2}}^{=0}\right) + \rho\overbrace{g_x}^{g\sin\theta} \tag{8.VIII.a}$$

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[26] German mechanical engineer, Ernst Kraft Wilhelm Nusselt born November 25, 1882 September 1, 1957 in Munchen

[27] Extensive discussion can be found in this author master thesis. Comprehensive discussion about this problem can be found this author Master thesis.

The first term of the acceleration is zero because the flow is in a steady state. The first term of the convective acceleration is zero under the assumption of this example flow is fully developed and hence not a function of $x$ (nothing to be "improved"). The second and the third terms in the convective acceleration are zero because the velocity at that direction is zero ($U_y = U_z = 0$). The pressure is almost constant along the $x$ coordinate. As it will be shown later, the pressure loss in the gas phase (mostly air) is negligible. Hence the pressure at the gas phase is almost constant hence the pressure at the interface in the liquid is constant. The surface has no curvature and hence the pressure at liquid side similar to the gas phase and the only change in liquid is in the $y$ direction. Fully developed flow means that the first term of the velocity Laplacian is zero ($\frac{\partial U_x}{\partial x} \equiv 0$). The last term of the velocity Laplacian is zero because no velocity in the $z$ direction.

Thus, equation (8.VIII.a) is reduced to

$$0 = \mu \frac{\partial^2 U_x}{\partial y^2} + \rho\, g \sin\theta \tag{8.VIII.b}$$

With boundary condition of "no slip" at the bottom because the large scale and steady state

$$U_x(y = 0) = 0 \tag{8.VIII.c}$$

The boundary at the interface is simplified to be

$$\left.\frac{\partial U_x}{\partial y}\right|_{y=0} = \tau_{air}\,(\sim 0) \tag{8.VIII.d}$$

If there is additional requirement, such a specific velocity at the surface, the governing equation can not be sufficient from the mathematical point of view. Integration of equation (8.VIII.b) yields

$$\frac{\partial U_x}{\partial y} = \frac{\rho}{\mu}\, g \sin\theta\, y + c_1 \tag{8.VIII.e}$$

The integration constant can be obtain by applying the condition (8.VIII.d) as

$$\tau_{air} = \mu \left.\frac{\partial U_x}{\partial y}\right|_{h} = -\rho\, g \sin\theta\, \overbrace{h}^{y} + c_1\, \mu \tag{8.VIII.f}$$

Solving for $c_1$ results in

$$c_1 = \frac{\tau_{air}}{\mu} + \frac{1}{\underbrace{\nu}_{\frac{\mu}{\rho}}}\, g\, \sin\theta\, h \tag{8.VIII.g}$$

The second integration applying the second boundary condition yields $c_2 = 0$ results in

$$U_x = \frac{g\, \sin\theta}{\nu} \left(2\, y\, h - y^2\right) - \frac{\tau_{air}}{\mu} \tag{8.VIII.h}$$

When the shear stress caused by the air is neglected, the velocity profile is

$$U_x = \frac{g\,\sin\theta}{\nu}\left(2\,h\,y - y^2\right) \tag{8.VIII.i}$$

The flow rate per unit width is

$$\frac{Q}{W} = \int_A U_x dA = \int_0^h \left(\frac{g\,\sin\theta}{\nu}\left(2\,h\,y - y^2\right) - \frac{\tau_{air}}{\mu}\right) dy \tag{8.VIII.j}$$

Where $W$ here is the width into the page of the flow. Which results in

$$\frac{Q}{W} = \frac{g\,\sin\theta}{\nu}\frac{2\,h^3}{3} - \frac{\tau_{air}\,h}{\mu} \tag{8.VIII.k}$$

The average velocity is then

$$\overline{U_x} = \frac{\dfrac{Q}{W}}{h} = \frac{g\,\sin\theta}{\nu}\frac{2\,h^2}{3} - \frac{\tau_{air}}{\mu} \tag{8.VIII.l}$$

Note the shear stress at the interface can be positive or negative and hence can increase or decrease the flow rate and the averaged velocity.

End Solution

In the following following example the issue of driving force of the flow through curved interface is examined. The flow in the kerosene lamp is depends on the surface tension. The flow surface is curved and thus pressure is not equal on both sides of the interface.

Example 8.9:

*A simplified flow version the kerosene lump is of liquid moving up on a solid core. Assume that radios of the liquid and solid core are given and the flow is at steady state. Calculate the minimum shear stress that required to operate the lump (alternatively, the maximum height).*

## 8.7.1 Interfacial Instability

In Example 8.8 no requirement was made as for the velocity at the interface (the upper boundary). The vanishing shear stress at the interface was the only requirement was applied. If the air is considered two governing equations must be solved one for the air (gas) phase and one for water (liquid) phase. Two boundary conditions must be satisfied at the interface. For the liquid, the boundary condition of "no

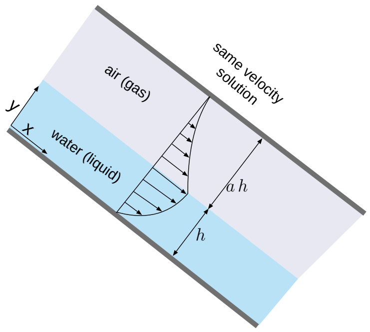

*Fig. -8.21. Flow of liquid in partially filled duct.*

slip" at the bottom surface of liquid must be satisfied. Thus, there is total of three boundary conditions[28] to be satisfied. The solution to the differential governing equations provides only two constants. The second domain (the gas phase) provides another equation with two constants but again three boundary conditions need to satisfied. However, two of the boundary conditions for these equations are the identical and thus the six boundary conditions are really only 4 boundary conditions.

The governing equation solution[29] for the gas phase ($h \geq y \geq a\,h$) is

$$U_{xg} = \frac{g\sin\theta}{2\,\nu_g}\,y^2 + c_1\,y + c_2 \tag{8.151}$$

Note, the constants $c_1$ and $c_2$ are dimensional which mean that they have physical units ($c_1 \longrightarrow [1/sec]$ The governing equation in the liquid phase ($0 \geq y \geq h$) is

$$U_{x\ell} = \frac{g\sin\theta}{2\,\nu_\ell}\,y^2 + c_3\,y + c_4 \tag{8.152}$$

The gas velocity at the upper interface is vanished thus

$$U_{xg}\left[(1+a)\,h\right] = 0 \tag{8.153}$$

At the interface the "no slip" condition is regularly applied and thus

$$U_{xg}(h) = U_{x\ell}(h) \tag{8.154}$$

Also at the interface (a straight surface), the shear stress must be continuous

$$\mu_g \frac{\partial U_{xg}}{\partial y} = \mu_\ell \frac{\partial U_{x\ell}}{\partial y} \tag{8.155}$$

Assuming "no slip" for the liquid at the bottom boundary as

$$U_{x\ell}(0) = 0 \tag{8.156}$$

The boundary condition (8.153) results in

$$0 = \frac{g\,\sin\theta}{2\,\nu_g}\,h^2\,(1+a)^2 + c_1\,h\,(1+a) + c_2 \tag{8.157}$$

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[28] The author was hired to do experiments on thin film (gravity flow). These experiments were to study the formation of small and big waves at the interface. The phenomenon is explained by the fact that there is somewhere instability which is transferred into the flow. The experiments were conducted on a solid concrete laboratory and the flow was in a very stable system. No matter how low flow rate was small and big occurred. This explanation bothered this author, thus current explanation was developed to explain the wavy phenomenon occurs.

[29] This equation results from double integrating of equation (8.VIII.b) and subtitling $\nu = \mu/\rho$.

The same can be said for boundary condition (8.156) which leads

$$c_4 = 0 \tag{8.158}$$

Applying equation (8.155) yields

$$\overbrace{\frac{\mu_g}{\nu_g}}^{\rho_g} g\,\sin\theta\, h + c_1\,\mu_g = \overbrace{\frac{\mu_\ell}{\nu_\ell}}^{\rho_\ell} g\,\sin\theta\, h + c_3\,\mu_\ell \tag{8.159}$$

Combining boundary conditions equation(8.154) with (8.157) results in

$$\frac{g\sin\theta}{2\,\nu_g}\, h^2 + c_1\, h + c_2 = \frac{g\sin\theta}{2\,\nu_\ell}\, h^2 + c_3\, h \tag{8.160}$$

*Advance material can be skipped*

The solution of equation (8.157), (8.159) and (8.160) is obtained by computer algebra (see in the code) to be

$$\begin{aligned}
c_1 &= -\frac{\sin\theta\,\left(g\,h\,\rho_g\,\left(2\,\rho_g\,\nu_\ell\,\rho_\ell+1\right)+a\,g\,h\,\nu_\ell\right)}{\rho_g\,\left(2\,a\,\nu_\ell+2\,\nu_\ell\right)}\\
c_2 &= \frac{\sin\theta\,\left(g\,h^2\,\rho_g\,\left(2\,\rho_g\,\nu_\ell\,\rho_\ell+1\right)-g\,h^2\,\nu_\ell\right)}{2\,\rho_g\,\nu_\ell}\\
c_3 &= \frac{\sin\theta\,\left(g\,h\,\rho_g\,\left(2\,a\,\rho_g\,\nu_\ell\,\rho_\ell-1\right)-a\,g\,h\,\nu_\ell\right)}{\rho_g\,\left(2\,a\,\nu_\ell+2\,\nu_\ell\right)}
\end{aligned} \tag{8.161}$$

*End Advance material*

When solving this kinds of mathematical problem the engineers reduce it to minimum amount of parameters to reduce the labor involve. So equation (8.157) transformed by some simple rearrangement to be

$$\left(1+a\right)^2 = \overbrace{\frac{2\,\nu_g\,c_1}{g\,h\,\sin\theta}}^{C_1} + \overbrace{\frac{2\,c_2\,\nu_g}{g\,h^2\,\sin\theta}}^{C_2} \tag{8.162}$$

And equation (8.159)

$$1 + \overbrace{\frac{\nu_g\,c_1}{g\,h\,\sin\theta}}^{\frac{1}{2}\,C_1} = \frac{\rho_\ell}{\rho_g} + \overbrace{\frac{\mu_\ell\,\nu_g\,c_3}{\mu_g\,g\,h\,\sin\theta}}^{\frac{1}{2}\,\frac{\mu_\ell}{\mu_g}\,C_3} \tag{8.163}$$

and equation (8.160)

$$1 + \frac{2\,\nu_g\,\cancel{h}\,c_1}{h^{\cancel{2}}\,g\,\sin\theta} + \frac{2\,\nu_g\,c_2}{h^2\,g\sin\theta} = \frac{\nu_g}{\nu_\ell} + \frac{2\,\nu_g\,\cancel{h}\,c_3}{g\,h^{\cancel{2}}\,\sin\theta} \tag{8.164}$$

Or rearranging equation (8.164)

$$\frac{\nu_g}{\nu_\ell} - 1 = \overbrace{\frac{2\,\nu_g\,c_1}{h\,g\,\sin\theta}}^{C_1} + \overbrace{\frac{2\,\nu_g\,c_2}{h^2\,g\sin\theta}}^{C_2} - \overbrace{\frac{2\,\nu_g\;c_3}{g\,h\,\sin\theta}}^{C_3} \tag{8.165}$$

This presentation provide similarity and it will be shown in the Dimensional analysis chapter better physical understanding of the situation. Equation (8.162) can be written as

$$\left(1+a\right)^2 = C_1 + C_2 \tag{8.166}$$

Further rearranging equation (8.163)

$$\frac{\rho_\ell}{\rho_g} - 1 = \frac{C_1}{2} - \frac{\mu_\ell}{\mu_g}\,\frac{C_3}{2} \tag{8.167}$$

and equation (8.165)

$$\frac{\nu_g}{\nu_\ell} - 1 = C_1 + C_2 - C_3 \tag{8.168}$$

This process that was shown here is referred as non–dimensionalization[30]. The ratio of the dynamics viscosity can be eliminated from equation (8.168) to be

$$\frac{\mu_g}{\mu_\ell}\,\frac{\rho_\ell}{\rho_g} - 1 = C_1 + C_2 - C_3 \tag{8.169}$$

The set of equation can be solved for the any ratio of the density and dynamic viscosity. The solution for the constant is

$$C_1 = \frac{\rho_g}{\rho_\ell} - 2 + a^2 + 2\,a\frac{\mu_g}{\mu_\ell} + 2\frac{\mu_g}{\mu_\ell} \tag{8.170}$$

$$C_2 = \frac{-\dfrac{\mu_g}{\mu_\ell}\,\dfrac{\rho_\ell}{\rho_g} + a\left(2\,\dfrac{\mu_g}{\mu_\ell} - 2\right) + 3\,\dfrac{\mu_g}{\mu_\ell} + a^2\left(\dfrac{\mu_g}{\mu_\ell} - 1\right) - 2}{\dfrac{\mu_g}{\mu_\ell}} \tag{8.171}$$

$$C_3 = -\frac{\mu_g}{\mu_\ell}\,\frac{\rho_\ell}{\rho_g} + a^2 + 2\,a + 2 \tag{8.172}$$

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[30] Later it will be move to the Dimensional Chapter

The two different fluids[31] have flow have a solution as long as the distance is finite reasonable similar. What happen when the lighter fluid, mostly the gas, is infinite long. This is one of the source of the instability at the interface. The boundary conditions of flow with infinite depth is that flow at the interface is zero, flow at infinite is zero. The requirement of the shear stress in the infinite is zero as well. There is no way obtain one dimensional solution for such case and there is a component in the $y$ direction. Combining infinite size domain of one fluid with finite size on the other one side results in unstable interface.

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[31] This topic will be covered in dimensional analysis in more extensively. The point here the understanding issue related to boundary condition not per se solution of the problem.