Note:

CHAPTER 10: DIMANALYSIS

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"Basics of Fluid Mechanics"

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CHAPTER 9

Dimensional Analysis

This chapter is dedicated to my adviser, Dr. E.R.G. Eckert. *Genick Bar-Meir*

9.1 Introductory Remarks

Dimensional analysis refers to techniques dealing with units or conversion to a unitless system. The definition of dimensional analysis is not consistent in the literature which span over various fields and times. Possible topics that dimensional analysis deals with are consistency of the units, change order of magnitude, applying from the old and known to unknown (see the Book of Ecclesiastes), and creation of group parameters without any dimensions. In this chapter, the focus is on the applying the old to unknown as different scales and the creation of dimensionless groups. These techniques gave birth to dimensional parameters which have a great scientific importance. Since the 1940s[1], the dimensional analysis is taught and written in all fluid mechanics textbooks. The approach or the technique used in these books is referred to as Buckingham–π–theory. The π–theory was coined by Buckingham. However, there is another technique which is referred to in the literature as the Nusselt's method. Both these methods attempt to reduce the number of parameters which affect the problem and reduce the labor in solving the problem. The key in these techniques lays in the fact of consistency of the dimensions of any possible governing equation(s) and the fact that some dimensions are reoccurring. The Buckingham–π goes further and no equations are solved and even no knowledge about these equations is required. In Buckingham's technique only the

[1] The history of dimensional analysis is complex. Several scientists used this concept before Buckingham and Nusselt (see below history section). Their work culminated at the point of publishing the paper Buckingham's paper and independently constructed by Nusselt. It is interesting to point out that there are several dimensionless numbers that bear Nusselt and his students name, Nusselt number, Schmidt number, Eckert number. There is no known dimensionless number which bears Buckingham name. Buckingham's technique is discussed and studied in Fluid Mechanics while almost completely ignored by Heat and Mass Transfer researchers and their classes. Furthermore, in many advance fluid mechanics classes Nusselt's technique is used and Buckingham's technique is abandoned. Perhaps this fact can be attributed to tremendous influence Nusselt and his students had on the heat transfer field. Even, this author can be accused for being bias as the Eckert's last student. However, this author observed that Nusselt's technique is much more effective as it will demonstrated later.

dimensions or the properties of the problem at hand are analyzed. This author is aware of only a single class of cases were Buckingham's methods is useful and or can solve the problem namely the pendulum class problem (and similar).

The dimensional analysis was independently developed by Nusselt and improved by his students/co workers (Schmidt, Eckert) in which the governing equations are used as well. Thus, more information is put into the problem and thus a better understanding on the dimensionless parameters is extracted. The advantage or disadvantage of these similar methods depend on the point of view. The Buckingham–π technique is simpler while Nusselt's technique produces a better result. Sometime, the simplicity of Buckingham's technique yields insufficient knowledge or simply becomes useless. When no governing equations are found, Buckingham's method has usefulness. It can be argued that these situations really do not exist in the Thermo–Fluid field. Nusselt's technique is more cumbersome but more precise and provide more useful information. Both techniques are discussed in this book. The advantage of the Nusselt's technique are: a) compact presentation, b)knowledge what parameters affect the problem, c) easier to extent the solution to more general situations. In very complex problems both techniques suffer from in inability to provide a significant information on the effective parameters such multi-phase flow etc.

It has to be recognized that the dimensional analysis provides answer to what group of parameters affecting the problem and not the answer to the problem. In fact, there are fields in thermo–fluid where dimensional analysis, is recognized as useless. For example, the area of multiphase flows there is no solution based on dimensionless parameters (with the exception of the rough solution of Martinelli). In the Buckingham's approach it merely suggests the number of dimensional parameters based on a guess of all parameters affecting the problem. Nusselt's technique provides the form of these dimensionless parameters, and the relative relationship of these parameters.

9.1.1 Brief History

The idea of experimentation with a different, rather than the actual, dimension was suggested by several individuals independently. Some attribute it to Newton (1686) who coined the phrase of "great Principle of Similitude." Later, Maxwell a Scottish Physicist played a major role in establishing the basic units of mass, length, and time as building blocks of all other units. Another example, John Smeaton (8 June 1724–28 October 1792) was an English civil and mechanical engineer who study relation between propeller/wind mill and similar devices to the pressure and velocity of the driving forces.

Jean B. J. Fourier (1768-1830) first attempted to formulate the dimensional analysis theory. This idea was extend by William Froude (1810-1871) by relating the modeling of open channel flow and actual body but more importantly the relationship between drag of models to actual ships. While the majority of the contributions were done by thermo–fluid guys the concept of the equivalent or similar propagated to other fields. Aiméem Vaschy, a German Mathematical Physicist (1857–1899), suggested using similarity in electrical engineering and suggested the Norton circuit equivalence theorems. Rayleigh probably was the first one who used dimensional analysis (1872) to obtain the relationships between the physical quantities (see the question why the sky is blue story).

Osborne Reynolds (1842–1912) was the first to derive and use dimensionless parameters to analyze experimental data. Riabouchunsky[2] proposed of relating temperature by molecules velocity and thus creating dimensionless group with the byproduct of compact solution (solution presented in a compact and simple form).

Buckingham culminated the dimensional analysis and similitude and presented it in a more systematic form. In the about the same time (1915, Wilhelm Nusselt (November 25, 1882 – September 1, 1957), a German engineer, developed the dimensional analysis (proposed the principal parameters) of heat transfer without knowledge about previous work of Buckingham.

9.1.2 Theory Behind Dimensional Analysis

In chemistry it was recognized that there are fundamental elements that all the material is made from (the atoms). That is, all the molecules are made from a combination of different atoms. Similarly to this concept, it was recognized that in many physical systems there are basic fundamental units which can describe all the other dimensions or units in the system. For example, isothermal single component systems (which does not undergo phase change, temperature change and observed no magnetic or electrical effect) can be described by just basic four physical units. The units or dimensions are, time, length, mass, quantity of substance (mole). For example, the dimension or the units of force can be constructed utilizing Newton's second law i.e. mass times acceleration $\longrightarrow m\,a = M\,L/t^2$. Increase of degree of freedom, allowing this system to be non–isothermal will increase only by one additional dimension of temperature, θ. These five fundamental units are commonly the building blocks for most of the discussion in fluid mechanics (see Table of basic units 9.1).

Table -9.1. Basic Units of Two Common Systems

Standard System			Old System		
Name	Letter	Units	**Name**	Letter	Unis
Mass	M	$[kg]$	**Force**	F	$[N]$
Length	L	$[m]$	**Length**	L	$[m]$
Time	t	$[sec]$	**Time**	t	$[sec]$
Temperature	θ	$[^{\circ}\mathrm{C}]$	**Temperature**	T	$[^{\circ}\mathrm{C}]$
Additional Basic Units for Magnetohydrodynamics					

Continued on next page

[2]Riabouchunsky, Nature Vol 99 p. 591, 1915

Table -9.1. Basic Units of Two Common Systems (continue)

Standard System			Old System		
Name	Letter	Units	**Name**	Letter	Unis
Electric Current	A	$[A]mpere$	**Electric Current**	A	$[A]mpere$
Luminous Intensity	cd	$[cd]$ candle	**Luminous Intensity**	cd	$[cd]$ candle
Chemical Reactions					
Quantity of substance	$\mathfrak{M}$	mol	**Quantity of substance**	$\mathfrak{M}$	mol

The choice of these basic units is not unique and several books and researchers suggest a different choice of fundamental units. One common selection is substituting the mass with the force in the previous selection (F, t, L, mol, Temperature). This author is not aware of any discussion on the benefits of one method over the other method. Yet, there are situations in which first method is better than the second one while in other situations, it can be the reverse. In this book, these two selections are presented. Other selections are possible but not common and, at the moment, will not be discussed here.

Example 9.1:
What are the units of force when the basic units are: mass, length, time, temperature (M, L, t, θ)? What are the units of mass when the basic units are: force, length, time, temperature (F, L, t, T)? Notice the different notation for the temperature in the two systems of basic units. This notation has no significance but for historical reasons remained in use.

Solution

These two systems are related as the questions are the reversed of each other. The connection between the mass and force can be obtained from the simplified Newton's second law $F = m\,a$ where F is the force, m is the mass, and a is the acceleration. Thus, the units of force are

$$F = \frac{M\,L}{t^2} \tag{9.I.a}$$

For the second method the unit of mass are obtain from Equation (9.I.a) as

$$M = \frac{F\,t^2}{L} \tag{9.I.b}$$

End Solution

The number of fundamental or basic dimensions determines the number of the combinations which affect the physical[3] situations. The dimensions or units which affect the problem at hand can be reduced because these dimensions are repeating or reoccurring. The Buckingham method is based on the fact that all equations must be consistent with their units. That is the left hand side and the right hand side have to have the same units. Because they have the same units the equations can be divided to create unitless equations. This idea alludes to the fact that these unitless parameters can be found without any knowledge of the governing equations. Thus, the arrangement of the effecting parameters in unitless groups yields the affecting parameters. These unitless parameters are the dimensional parameters. The following trivial example demonstrates the consistency of units

Example 9.2:
Newton's equation has two terms that related to force $F = m\,a + \dot{m}U$. Where F is force, m is the mass, a is the acceleration and dot above $\dot{m}$ indicating the mass derivative with respect to time. In particular case, this equation get a form of

$$F = m\,a + 7 \tag{9.II.a}$$

where 7 represent the second term. What are the requirement on equation (9.II.a)*?*

SOLUTION

Clearly, the units of $[F]$, $m\,a$ and 7 have to be same. The units of force are $[N]$ which is defined by first term of the right hand side. The same units force has to be applied to 7 thus it must be in $[N]$.

End Solution

9.1.3 Dimensional Parameters Application for Experimental Study

The solutions for any situations which are controlled by the same governing equations with same boundary conditions regardless of the origin the equation. The solutions are similar or identical regardless to the origin of the field no matter if the field is physical, or economical, or biological. The Buckingham's technique implicitly suggested that since the governing equations (in fluid mechanics) are essentially are the same, just knowing the parameters is enough the identify the problem. This idea alludes to connections between similar parameters to similar solution. The non–dimensionalization i.e. operation of reducing the number affecting parameters, has a useful by–product, the analogy in other words, the solution by experiments or other cases. The analogy or similitude refers to understanding one phenomenon from the study of another phenomenon. This technique is employed in many fluid mechanics situations. For example, study of compressible flow (a flow where the density change plays a significant part) can be achieved

[3]The dimensional analysis also applied in economics and other areas and the statement should reflect this fact. However, this book is focused on engineering topics and other fields are not discussed.

by study of surface of open channel flow. The compressible flow is also similar to traffic on the highway. Thus for similar governing equations if the solution exists for one case it is a solution to both cases.

The analogy can be used to conduct experiment in a cheaper way and/or a safer way. Experiments in different scale than actual dimensions can be conducted for cases where the actual dimensions are difficult to handle. For example, study of large air planes can done on small models. On the other situations, larger models are used to study small or fast situations. This author believes that at the present the Buckingham method has extremely limited use for the real world and yet this method is presented in the classes on fluid mechanics. Thus, many examples on the use of this method will be presented in this book. On the other hand, Nusselt's method has a larger practical use in the real world and therefore will be presented for those who need dimensional analysis for the real world. Dimensional analysis is useful also for those who are dealing with the numerical research/calculation. This method supplement knowledge when some parameters should be taken into account and why.

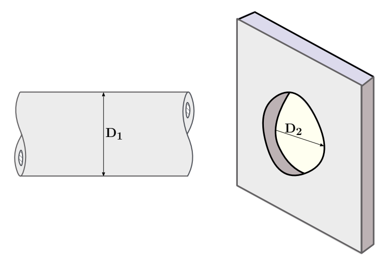

Fig. -9.1. Fitting rod into a hole.

Fitting a rod into a circular hole (see Figure 9.1) is an example how dimensional analysis can be used. To solve this problem, it is required to know two parameters; 1) the rode diameter and 2) the diameter of the hole. Actually, it is required to have only one parameter, the ratio of the rode diameter to the hole diameter. The ratio is a dimensionless number and with this number one can tell that for a ratio larger than one, the rode will not enter the hole; and a ratio smaller than one, the rod is too small. Only when the ratio is equal to one, the rode is said to be fit. This presentation allows one to draw or present the situation by using only one coordinate, the radius ratio. Furthermore, if one wants to deal with tolerances, the dimensional analysis can easily be extended to say that when the ratio is equal from 0.99 to 1.0 the rode is fitting, and etc. If one were to use the two diameters description, further significant information will be needed. In the preceding simplistic example, the advantages are minimal. In many real problems this approach can remove clattered views and put the problem into focus. Throughout this book the reader will notice that the systems/equations in many cases are converted to a dimensionless form to augment understanding.

9.1.4 The Pendulum Class Problem

The only known problem that dimensional analysis can solve (to some degree) is the pendulum class problem. In this section several examples of the pendulum type problem are presented. The first example is the classic Pendulum problem.

Example 9.3:

Derive the relationship for the gravity [g], frequency [ω] and length of pendulum [ℓ]. Assume that no other parameter including the mass affects the problem. That is, the relationship can be expressed as

$$\omega = f\left(\ell, g\right) \tag{9.III.a}$$

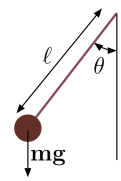

Fig. -9.2. Figure for example 9.3

Notice in this problem, the real knowledge is provided, however in the real world, this knowledge is not necessarily given or known. Here it is provided because the real solution is already known from standard physics classes.[4]

<u>Solution</u>

The solution technique is based on the assumption that the indexical form is the appropriate form to solve the problem. The Indexical form

$$\omega = C_1 \times \ell^a g^b \tag{9.III.b}$$

The solution functional complexity is limited to the basic combination which has to be in some form of multiplication of ℓ and g in some power. In other words, the multiplication of $\ell\, g$ have to be in the same units of the frequency units. Furthermore, assuming, for example, that a trigonometric function relates ℓ and g and frequency. For example, if a $\sin$ function is used, then the functionality looks like $\omega = \sin(\ell\, g)$. From the units point of view, the result of operation not match i.e. $(sec \neq \sin\left(sec\right))$. For that reason the form in equation (9.III.b) is selected. To satisfy equation (9.III.b) the units of every term are examined and summarized the following table.

Table -9.2. Units of the Pendulum Parameters

Parameter	**Units**	**Parameter**	**Units**	**Parameter**	**Units**
ω	t^{-1}	ℓ	L^1	g	$L^1 t^{-2}$

Thus substituting of the Table 9.7 in equation (9.III.b) results in

$$t^{-1} = C_1 \left(L^1\right)^a \left(L^1\, t^{-2}\right)^b \Longrightarrow L^{a+b} t^{-2\,b} \tag{9.III.c}$$

after further rearrangement by multiply the left hand side by L^0 results in

$$L^0 t^{-1} = C\, L^{a+b} t^{-2\,b} \tag{9.III.d}$$

[4]The reader can check if the mass is assumed to affect the problem then, the result is different.

In order to satisfy equation (9.III.d), the following must exist

$$0 = a + b \quad and \quad -1 = \tfrac{-2}{b} \tag{9.III.e}$$

The solution of the equations (9.III.e) is $a = -1/2$ and $b = -1/2$. Thus, the solution is in the form of

$$\omega = C_1 \, \ell^{1/2} \, g^{-1/2} = C_1 \sqrt{\frac{g}{\ell}} \tag{9.III.f}$$

It can be observed that the value of C_1 is unknown. The pendulum frequency is known to be

$$\omega = \frac{1}{2\pi} \sqrt{\frac{g}{\ell}} \tag{9.III.g}$$

End Solution

What was found in this example is the form of the solution's equation and frequency. Yet, the functionality e.g. the value of the constant was not found. The constant can be obtained from experiment for plotting ω as the abscissa and $\sqrt{\ell/g}$ as ordinate.

According to some books and researchers, this part is the importance of the dimensional analysis. It can be noticed that the initial guess merely and actually determine the results. If, however, the mass is added to considerations, a different result will be obtained. If the guess is relevant and correct then the functional relationship can be obtained by experiments.

9.2 Buckingham–π–Theorem

All the physical phenomena that is under the investigation have n physical effecting parameters such that

$$F_1(q_1, q_2, q_3, \cdots, q_n) = 0 \tag{9.1}$$

where q_i is the "i" parameter effecting the problem. For example, study of the pressure difference created due to a flow in a pipe is a function of several parameters such

$$\Delta P = f(L, D, \mu, \rho, U) \tag{9.2}$$

In this example, the chosen parameters are not necessarily the most important parameters. For example, the viscosity, μ can be replaced by dynamic viscosity, ν. The choice is made normally as the result of experience and it can be observed that ν is a function of μ and ρ. Finding the important parameters is based on "good fortune" or perhaps intuition. In that case, a new function can be defined as

$$F(\Delta P, L, D, \mu, \rho, U) = 0 \tag{9.3}$$

Again as stated before, the study of every individual parameter will create incredible amount of data. However, Buckingham's[5] methods suggested to reduce the number of

[5] E. Buckingham, "Model Experiments and the Forms of Empirical Equations," Transactions of the American Society of Mechanical Engineers, Vol. 37, 1915.

parameters. If independent parameters of same physical situation is m thus in general it can be written as

$$F_2(\pi_1, \pi_2, \pi_3, \cdots, \pi_m) = 0 \tag{9.4}$$

If there are n variables in a problem and these variables contain m primary dimensions (for example M, L, T), then the equation relating all the variables will have (n-m) dimensionless groups.

There are 2 conditions on the dimensionless parameters:

1. Each of the fundamental dimensions must appear in at least one of the m variables
2. It must not be possible to form a dimensionless group from one of the variables within a recurring set. A recurring set is a group of variables forming a dimensionless group.

In the case of the pressure difference in the pipe (Equation (9.3)) there are 6 variables or $n = 6$. The number of the fundamental dimensions is 3 that is $m = 3$ ([M], [L], [t]) The choice of fundamental or basic units is arbitrary in that any construction of these units is possible. For example, another combination of the basic units is time, force, mass is a proper choice. According to Buckingham's theorem the number of dimensionless groups is $n - m = 6 - 3 = 3$. It can be written that one dimensionless parameter is a function of two other parameters such as i

$$\pi_1 = f(\pi_2, \pi_3) \tag{9.5}$$

If indeed such a relationship exists, then, the number of parameters that control the problem is reduced and the number of experiments that need to be carried is considerably smaller. Note, the π–theorem does not specify how the parameters should be selected nor what combination is preferred.

9.2.1 Construction of the Dimensionless Parameters

In the construction of these parameters it must be realized that every dimensionless parameters has to be independent. The meaning of independent is that one dimensionless parameter is not a multiply or a division of another dimensional parameter. In the above example there are three dimensionless parameters which required of at least one of the physical parameter per each dimensionless parameter. Additionally, to make these dimensionless parameters independent they cannot be multiply or division of each other.

For the pipe problem above, ℓ and D have the same dimension and therefore both cannot be chosen as they have the same dimension. One possible combination is of D, U and ρ are chosen as the recurring set. The dimensions of these physical variables are: $D = [L^1]$, velocity of $U = [L\,t^{-1}]$ and density as $\rho = [M\,L^{-3}]$. Thus, the first term D can provide the length, $[L]$, the second term, U, can provide the time $[t]$, and the third term, ρ can provide the mass $[M]$. The fundamental units, L, t, and M are length, time and mass respectively. The fundamental units can be written in terms of the physical units. The first term L is the described by D with the units of $[L]$. The time, $[t]$, can be expressed by D/U. The mass, $[M]$, can be expressed by $\rho\, D^3$. Now the dimensionless groups can be constructed by looking at the remaining physical parameters, ΔP, D and μ. The pressure difference, ΔP, has dimensions of $[M\, L^{-1}\, t^{-2}]$ Therefore, $\Delta P\, M^{-1}\, L\, t^2$ is a dimensionless quantity and these values were calculated just above this line. Thus, the first dimensionless group is

$$\pi_1 = \overbrace{\Delta P}^{[M\,L^{-1}\,t^{-2}]} \overbrace{\frac{1}{\rho\, D^3}}^{[M^{-1}]} \overbrace{D}^{[L]} \overbrace{\frac{D^2}{U^2}}^{[t^2]} = \overbrace{\frac{\Delta P}{\rho\, U^2}}^{unitless} \tag{9.6}$$

The second dimensionless group (using D) is

$$\pi_2 = \overbrace{D}^{[L]} \overbrace{\ell^{-1}}^{[L^{-1}]} = \frac{D}{L} \tag{9.7}$$

The third dimensionless group (using μ dimension of $[M\, L^1\, t^{-1}]$) and therefore dimensionless is

$$\pi_3 = \mu \overbrace{\frac{1}{D^3\, \rho}}^{[M^{-1}]} \overbrace{D}^{[L]} \overbrace{\frac{D}{U}}^{[t]} = \frac{\mu}{D\, U\, \rho} \tag{9.8}$$

This analysis is not unique and there can be several other possibilities for selecting dimensionless parameters which are "legitimately" correct for this approach.

There are roughly three categories of methods for obtaining the dimensionless parameters. The first one solving it in one shot. This method is simple and useful for a small number of parameters. Yet this method becomes complicated for large number of parameters. The second method, some referred to as the building blocks method, is described above. The third method is by using dimensional matrix which is used mostly by mathematicians and is less useful for engineering purposes.

The second and third methods require to identification of the building blocks. These building blocks are used to construct the dimensionless parameters. There are several requirements on these building blocks which were discussed on page 287. The main point that the building block unit has to contain at least the basic or fundamental unit. This requirement is logical since it is a building block. The last method is mostly used by mathematicians which leads and connects to linear algebra. The fact that this method used is the hall mark that the material was written by mathematician. Here, this material will be introduced for completeness sake with examples and several terms associated with this technique.

9.2.2 Basic Units Blocks

In Thermo–Fluid science there are several basic physical quantities which summarized in Table 9.1. In the table contains two additional physical/basic units that appear in magnetohydrodynamics (not commonly use in fluid mechanics). Many (almost all) of the engineering dimensions used in fluid mechanics can be defined in terms of the four basic physical dimensions M, L ,t and θ. The actual basic units used can be S.I. such as kilograms, meters, seconds and Kelvins/Celsius or English system or any other system. In using basic new basic physical units, M, L, t, and θ or the old system relieves the discussion from using particular system measurements. The density, for example, units are $Mass/Length^3$ and in the new system the density will be expressed as M/L^3 while in S.I. kg/m^3 and English system it $slug/ft^3$. A common unit used in Fluid Mechanics is the Force, which expressed in SI as Newton $[N]$. The Newton defined as a force which causes a certain acceleration of a specific mass. Thus, in the new system the force it will be defined as $M\,L\,t^{-2}$. There are many parameters that contains force which is the source reason why the old (or alternative) system use the force instead the mass.

There many physical units which are dimensionless by their original definition. Examples to "naturally" being dimensionless are the angle, strains, ratio of specific heats, k, friction coefficient, f and ratio of lengths. The angle represented by a ratio of two sides of a triangle and therefor has no units nor dimensions. Strain is a ratio of the change of length by the length thus has no units.

Quantities used in engineering can be reduced to six basic dimensions which are presented in Table 9.1. The last two are not commonly used in fluid mechanics and temperature is only used sometimes. Many common quantities are presented in the following Table 9.3.

Table -9.3. Physical units for two common systems. Note the second (time) in large size units appear as "s" while in small units as "$sec.$"

Standard System			Old System		
Name	Letter	Units	**Name**	Letter	Unis
Area	L^2	$[m^2]$	**Area**	L^2	$[m^2]$
Volume	L^3	$[m^3]$	**Volume**	L^3	$[m^3]$
Angular velocity	$\dfrac{1}{t}$	$\left[\frac{1}{sec}\right]$	**Angular velocity**	$\dfrac{1}{t}$	$\left[\frac{1}{sec}\right]$
Acceleration	$\dfrac{L}{t^2}$	$\left[\frac{m}{sec^2}\right]$	**Acceleration**	$\dfrac{L}{t^2}$	$\left[\frac{m}{sec^2}\right]$
Angular acceleration	$\dfrac{1}{t^2}$	$\left[\frac{1}{sec^2}\right]$	**Angular acceleration**	$\dfrac{1}{t^2}$	$\left[\frac{1}{sec^2}\right]$
Force	$\dfrac{M\,L}{t^2}$	$\left[\frac{kg\,m}{sec^2}\right]$	**Mass**	$\frac{F\,t^2}{L}$	$\left[\frac{N\,s}{m}\right]$
Density	$\dfrac{M}{L^3}$	$\left[\frac{kg}{m^3}\right]$	**Density**	$\dfrac{F\,t^2}{L^4}$	$\left[\frac{kg}{m^3}\right]$

Continued on next page

Table -9.3. Basic Units of Two Common System (continue)

Standard System			Old System		
Name	Letter	Units	**Name**	Letter	Unis
Momentum	$\frac{M\,L}{t}$	$\left[\frac{kg\,m}{sec}\right]$	**Momentum**	$F\,t$	$[N\,sec]$
Angular Momentum	$\frac{M\,L^2}{t}$	$\left[\frac{kg\,m^2}{sec}\right]$	**Angular Momentum**	$L\,F\,t$	$[m\,N\,s]$
Torque	$\frac{M\,L^2}{t^2}$	$\left[\frac{kg\,m}{sec^2}\right]$	**Torque**	$L\,F$	$[m\,N]$
Absolute Viscosity	$\frac{M}{L^1\,t^1}$	$\left[\frac{kg}{m\,s}\right]$	**Absolute Viscosity**	$\frac{t\,F}{L^2}$	$\left[\frac{N\,s}{m^2}\right]$
Kinematic Viscosity	$\frac{L^2}{t^1}$	$\left[\frac{m^2}{sec}\right]$	**Kinematic Viscosity**	$\frac{L^2}{t}$	$\left[\frac{m^3}{sec}\right]$
Volume flow rate	$\frac{L^3}{t^1}$	$[sec]$	**Volume flow rate**	$\frac{L^3}{t^1}$	$\left[\frac{m^3}{sec}\right]$
Mass flow rate	$\frac{M}{t^1}$	$\left[\frac{kg}{sec}\right]$	**Mass flow rate**	$\frac{F\,t}{L^1}$	$\left[\frac{N\,s}{m}\right]$
Pressure	$\frac{M}{L\,t^2}$	$\left[\frac{kg}{m\,sec}\right]$	**Pressure**	$\frac{F}{L^2}$	$\left[\frac{N}{m^2}\right]$
Surface Tension	$\frac{M}{t^2}$	$\left[\frac{kg}{sec^2}\right]$	**Surface Tension**	$\frac{F}{L}$	$\left[\frac{N}{m}\right]$
Work or Energy	$\frac{M\,L^2}{t^2}$	$\left[\frac{kg\,m^2}{sec^2}\right]$	**Work or Energy**	$F\,L$	$[N\,m]$
Power	$\frac{M\,L^2}{t^3}$	$\left[\frac{kg\,m^2}{sec^3}\right]$	**Power**	$\frac{F\,L}{t^1}$	$\left[\frac{N\,m}{sec}\right]$
Thermal Conductivity	$\frac{M\,L^2}{t^3\,\theta}$	$\left[\frac{kg\,m^2}{s^2\,K}\right]$	**Thermal Conductivity**	$\frac{F}{t\,T}$	$\left[\frac{N}{m\,K}\right]$
Specific Heat	$\frac{L^2\,\theta^2}{t^2}$	$\left[\frac{m^2}{s^2\,K}\right]$	**Specific Heat**	$\frac{L^2\,T^2}{t^2}$	$\left[\frac{m^2}{s^2\,K}\right]$
Entropy	$\frac{M\,L^2}{t^2\,\theta}$	$\left[\frac{kg\,m^2}{s^2\,K}\right]$	**Entropy**	$\frac{F\,L^2}{T}$	$\left[\frac{kg\,m^2}{s^2\,K}\right]$
Specific Entropy	$\frac{L^2}{t^2\,\theta}$	$\left[\frac{m^2}{s^2\,K}\right]$	**Specific Entropy**	$\frac{L^2}{t^2\,T}$	$\left[\frac{m^2}{s^2\,K}\right]$

Continued on next page

Table -9.3. Basic Units of Two Common System (continue)

Standard System			Old System		
Name	Letter	Units	**Name**	Letter	Unis
Molar Specific Entropy	$\frac{L^2}{t^2\,\theta}$	$\left[\frac{kg\,m^2}{s^2\,K\,mol}\right]$	**Molar Specific Entropy**	$\frac{L^2}{T\,t^2}$	$\left[\frac{kg\,m^2}{s^2\,K\,mol}\right]$
Enthalpy	$\frac{M\,L^2}{t^2}$	$\left[\frac{kg\,m^2}{sec^2}\right]$	**Enthalpy**	$F\,L$	$[N\,m]$
Specific Enthalpy	$\frac{M^2}{t^2}$	$\left[\frac{m^2}{sec^2}\right]$	**Specific Enthalpy**	$\frac{L^2}{t^2}$	$\left[\frac{m^2}{sec^2}\right]$
Thermodynam Force	$\frac{M\,L}{t^2\,\mathfrak{M}}$	$\left[\frac{kg\,m}{sec^2\,mol}\right]$	**Thermodynam Force**	$\frac{N}{\mathfrak{M}}$	$\left[\frac{m^2}{sec^2}\right]$
Catalytic Activity	$\frac{\mathfrak{M}}{t}$	$\left[\frac{mol}{sec}\right]$	**Catalytic Activity**	$\frac{\mathfrak{M}}{t}$	$\left[\frac{mol}{sec}\right]$
heat transfer rate	$\frac{M\,L^2}{t^3}$	$\left[\frac{kg\,m^2}{sec^2}\right]$	**heat transfer rate**	$\frac{L\,F}{t}$	$\left[\frac{m\,N}{sec}\right]$

9.2.3 Implementation of Construction of Dimensionless Parameters

9.2.3.1 One Shot Method: Constructing Dimensionless Parameters

In this method, the solution is obtained by assigning the powers to the affecting variables. The results are used to compare the powers on both sides of the equation. Several examples are presented to demonstrate this method.

Example 9.4:

An infinite cylinder is submerged and exposed to an external viscous flow. The researcher intuition suggests that the resistance to flow, R is a function of the radius r, the velocity U, the density, ρ, and the absolute viscosity μ. Based on this limited information construct a relationship of the variables, that is

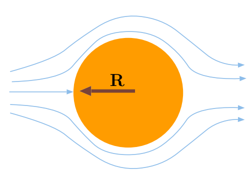

Fig. -9.3. Resistance of infinite cylinder.

$$R = f(r, U, \rho, \mu) \tag{9.IV.a}$$

Solution

The functionality should be in a form of

$$R = f\left(r^a\,U^b\,\rho^c\mu^d\right) \tag{9.IV.b}$$

The units of the parameters are provided in Table 9.3. Thus substituting the data from the table into equation (9.IV.b) results in

$$\frac{\overbrace{M\,L}^{R}}{t^2} = Constant\left(\overbrace{L}^{r}\right)^a \left(\overbrace{\frac{L}{t}}^{U}\right)^b \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^c \left(\overbrace{\frac{M}{L\,t}}^{\mu}\right)^d \tag{9.IV.c}$$

From equation (9.IV.c) the following requirements can be obtained

$$\begin{array}{lrcl} time, t & -2 & = & -b-d \\ mass, M & 1 & = & c+d \\ length, L & 1 & = & a+b-3c-d \end{array} \tag{9.IV.d}$$

In equations (9.IV.c) there are three equations and 4 unknowns. Expressing all the three variables in term of d to obtain

$$\begin{array}{rcl} a & = & 2-d \\ b & = & 2-d \\ c & = & 1-d \end{array} \tag{9.IV.e}$$

Substituting equation (9.IV.e) into equation (9.IV.c) results in

$$R = Constant\,r^{2-d}\,U^{2-d}\,\rho^{1-d}\,\mu^d = Constant\,\left(\rho\,U^2\,r^2\right)\left(\frac{\mu}{\rho\,U\,r}\right)^d \tag{9.IV.f}$$

Or rearranging equation yields

$$\frac{R}{\rho\,U^2\,r^2} = Constant\,\left(\frac{\mu}{\rho\,U\,r}\right)^d \tag{9.IV.g}$$

The relationship between the two sides in equation (9.IV.g) is related to the two dimensionless parameters. In dimensional analysis the functionality is not clearly defined by but rather the function of the parameters. Hence, a simple way, equation (9.IV.g) can be represented as

$$\frac{R}{\rho\,U^2\,r^2} = Constant\,f\left(\frac{\mu}{\rho\,U\,r}\right) \tag{9.IV.h}$$

where the power of d can be eliminated.

End Solution

An example of a ship[6] is be a typical example were more than one dimensionless is to constructed. Also introduction of dimensional matrix is presented.

Example 9.5:

The modern ship today is equipped with a propeller as the main propulsion mechanism. The thrust, T is known to be a function of the radius, r, the fluid density, ρ, relative velocity of the ship to the water, U, rotation speed, rpm or N, and fluid viscosity, μ. Assume that no other parameter affects the thrust, find the functionality of these parameters and the thrust.

Solution

The general solution under these assumptions leads to solution of

$$T = C\, r^a\, \rho^b\, U^c\, N^d\, \mu^e \tag{9.V.a}$$

It is convenient to arrange the dimensions and basic units in table. This table is referred in the literature as the Dimensional matrix.

Table -9.4. Dimensional matrix

	T	**r**	$\boldsymbol{\rho}$	**U**	**N**	$\boldsymbol{\mu}$
M	1	0	1	0	0	1
L	1	1	-3	1	0	-1
t	-2	0	0	-1	-1	-1

Using the matrix results in

$$M\, Lt^{-2} = L^a\ (Lt)^b \left(ML^{-3}\right)^c \left(t^{-t}\right)^d \left(ML^{-1}t^{-t}\right)^e \tag{9.V.b}$$

This matrix leads to three equations.

$$\begin{aligned} \text{Mass}, M \quad 1 = &\quad c + e \\ \text{Length}, L \quad 1 = &\quad a + b + -3c - e \\ \text{time}, t \quad -2 = &\quad -c - d - e \end{aligned} \tag{9.V.c}$$

[6]This author who worked as ship engineer during his twenties likes to present material related to ships.

The solution of this system is

$$
\begin{aligned}
a = &\quad 2 + d - e \\
b = &\quad 2 - d - e \\
c = &\quad 1 - e
\end{aligned}
\tag{9.V.d}
$$

Substituting the solution (9.V.d) into equation (9.V.a) yields

$$
T = C\, r^{(2+d-e)}\, \rho^{(2-d-e)}\, U^{(1-e)}\, N^d\, \mu^f \tag{9.V.e}
$$

Rearranging equation (9.V.e) provides

$$
T = C\, \rho\, U^2\, r^2 \left(\frac{\rho\, U\, r}{\mu}\right)^d \left(\frac{r\, N}{U}\right)^e \tag{9.V.f}
$$

From dimensional analysis point of view the units under the power d and e are dimensionless. Hence, in general it can be written that

$$
\frac{T}{\rho\, U^2\, r^2} = f\left(\frac{\rho\, U\, r}{\mu}\right) g\left(\frac{r\, N}{U}\right) \tag{9.V.g}
$$

where f and g are arbitrary functions to be determined in experiments. Note the rpm or N refers to the rotation in radian per second even though rpm refers to revolution per minute.

It has to be mentioned that these experiments have to constructed in such way that the initial conditions and the boundary conditions are somehow "eliminated." In practical purposes the thrust is a function of Reynolds number and several other parameters. In this example, a limited information is provided on which only Reynolds number with a additional dimensionless parameter is mentioned above.

End Solution

Example 9.6:

The surface wave is a small disturbance propagating in a liquid surface. Assume that this speed for a certain geometry is a function of the surface tension, σ, density, ρ, and the wave length of the disturbance (or frequency of the disturbance). The flow–in to the chamber or the opening of gate is creating a disturbance. The knowledge when this disturbance is important and is detected by with the time it traveled. The time control of this certain process is critical because the chemical kinetics. The calibration of the process was done with satisfactory results. Technician by mistake releases a chemical which reduces the surface tension by half. Estimate the new speed of the disturbance.

SOLUTION

In the problem the functional analysis was defined as

$$
U = f(\sigma,\, \rho,\, \lambda) \tag{9.VI.a}
$$

Equation (9.VI.a) leads to three equations as

$$\overbrace{\frac{L}{t}}^{U} = \left(\overbrace{\frac{M}{L^2}}^{\rho}\right)^a \left(\overbrace{\frac{M}{t^2}}^{\sigma}\right)^b \left(\overbrace{L}^{\lambda}\right)^c \tag{9.VI.b}$$

$$\begin{array}{rrr} \text{Mass}, M & a + b = & 0 \\ \text{Length}, L & -2a + c = & 1 \\ \text{time}, t & -2b = & -1 \end{array} \tag{9.VI.c}$$

The solution of equation set (9.VI.c) results in

$$U = \sqrt{\frac{\sigma}{\lambda\,\rho}} \tag{9.VI.d}$$

Hence reduction of the surface tension by half will reduce the disturbance velocity by $1/\sqrt{2}$.

End Solution

Example 9.7:

Eckert number represent the amount of dissipation. Alternative number represents the dissipation, could be constructed as

$$Diss = \frac{\mu \left(\dfrac{dU}{d\ell}\right)^2}{\dfrac{\rho\,U^2}{\dfrac{\ell}{U}}} = \frac{\mu \left(\dfrac{dU}{d\ell}\right)^2 \ell}{\rho\,U^3} \tag{9.VII.a}$$

Show that this number is dimensionless. What is the physical interpretation it could have? Flow is achieved steady state for a very long two dimensional channel where the upper surface is moving at speed, U_{up}, and lower is fix. The flow is pure Couette flow i.e. a linear velocity. Developed an expression for dissipation number using the information provided.

Solution

The nominator and denominator have to have the same units.

$$\overbrace{\frac{M}{\cancel{L}\,t}}^{\mu}\overbrace{\frac{\cancel{L^2}}{t^2\,\cancel{L^2}}}^{\left(\frac{dU}{d\ell}\right)^2}\overbrace{\cancel{L}}^{\ell} = \overbrace{\frac{M}{\cancel{L^3}}}^{\rho}\overbrace{\frac{\cancel{L^3}}{t^3}}^{U^3} \tag{9.VII.b}$$

$$\rightsquigarrow \quad \frac{M}{t^3} = \frac{M}{t^3}$$

The averaged velocity could be a represented (there are better methods or choices) of the energy flowing in the channel. The averaged velocity is $U/2$ and the velocity derivative is $dU/d\ell = constant = U/\ell$. With these value of the Diss number is

$$Diss = \frac{\mu \left(\dfrac{U}{\ell}\right)^2 \ell}{\rho \dfrac{U^3}{8}} = \frac{4\,\mu}{\rho\,\ell\,U} \tag{9.VII.c}$$

The results show that Dissipation number is not a function of the velocity. Yet, the energy lost is a function of the velocity square $E \propto Diss\,\mu\,U$.

End Solution

9.2.3.2 Building Blocks Method: Constructing Dimensional Parameters

Note, as opposed to the previous method, this technique allows one to find a single or several dimensionless parameters without going for the whole calculations of the dimensionless parameters.

Example 9.8:
Assume that the parameters that effects the centrifugal pumps are

Q	*Pump Flow rate*	rpm *or* N	*angular rotation speed*
D	*rotor diameter*	ρ	*liquid density (assuming liquid phase)*
B_T	*Liquid Bulk modulus*	μ	*liquid viscosity*
ϵ	*typical roughness of pump surface*	g	*gravity force (body force)*
ΔP	*Pressure created by the pump*		

Construct the functional relationship between the variables. Discuss the physical meaning of these numbers. Discuss which of these dimensionless parameters can be neglected as it is known reasonably.

Solution

The functionality can be written as

$$0 = f\left(D,\, N,\, \rho,\, Q,\, B_T,\, \mu,\, \epsilon,\, g,\, \Delta P\right) \tag{9.VIII.a}$$

The three basic parameters to be used are D [L], ρ [M], and N [t]. There are nine (9) parameters thus the number of dimensionless parameters is $9 - 3 = 6$. For simplicity the RPM will be denoted as N. The first set is to be worked on is Q, D, ρ, N as

$$\overbrace{\frac{L^3}{t}}^{Q} = \left(\overbrace{L}^{D}\right)^a \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^b \left(\overbrace{\frac{1}{t}}^{N}\right)^c \tag{9.VIII.b}$$

$$\left.\begin{array}{rrr} \text{Length}, L & a-3b = & 3 \\ \text{Mass}, M & b = & 0 \\ \text{time}, t & -c = & -1 \end{array}\right\} \Longrightarrow \pi_1 = \frac{Q}{N\,D^3} \tag{9.VIII.c}$$

For the second term B_T it follows

$$\overbrace{\frac{M}{L\,t^2}}^{B_T} = \left(\overbrace{L}^{D}\right)^a \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^b \left(\overbrace{\frac{1}{t}}^{N}\right)^c \tag{9.VIII.d}$$

$$\left.\begin{array}{rrr} \text{Mass}, M & b = & 1 \\ \text{Length}, L & a-3b = & -1 \\ \text{time}, t & -c = & -2 \end{array}\right\} \Longrightarrow \pi_2 = \frac{B_T}{\rho\,N^2\,D^2} \tag{9.VIII.e}$$

The next term, μ,

$$\overbrace{\frac{M}{L\,t}}^{\mu} = \left(\overbrace{L}^{D}\right)^a \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^b \left(\overbrace{\frac{1}{t}}^{N}\right)^c \tag{9.VIII.f}$$

$$\left.\begin{array}{rrr} \text{Mass}, M & b = & 1 \\ \text{Length}, L & a-3b = & -1 \\ \text{time}, t & -c = & -1 \end{array}\right\} \Longrightarrow \pi_3 = \frac{\rho\,N^2\,D^2}{\mu} \tag{9.VIII.g}$$

The next term, ϵ,

$$\overbrace{L}^{\epsilon} = \left(\overbrace{L}^{D}\right)^a \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^b \left(\overbrace{\frac{1}{t}}^{N}\right)^c \tag{9.VIII.h}$$

$$\left.\begin{array}{rrr} \text{Mass}, M & b = & 0 \\ \text{Length}, L & a-3b = & 1 \\ \text{time}, t & -c = & 0 \end{array}\right\} \Longrightarrow \pi_4 = \frac{\epsilon}{D} \tag{9.VIII.i}$$

The next term, g,

$$\overbrace{\frac{L}{t^2}}^{g} = \left(\overbrace{L}^{D}\right)^a \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^b \left(\overbrace{\frac{1}{t}}^{N}\right)^c \tag{9.VIII.j}$$

$$\left.\begin{array}{rrr} \text{Mass}, M & b = & 0 \\ \text{Length}, L & a - 3b = & 1 \\ \text{time}, t & -c = & -2 \end{array}\right\} \Longrightarrow \pi_5 = \frac{g}{D\,N^2} \tag{9.VIII.k}$$

The next term, ΔP, (similar to B_T)

$$\overbrace{\frac{L}{t^2}}^{\Delta P} = \left(\overbrace{L}^{D}\right)^a \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^b \left(\overbrace{\frac{1}{t}}^{N}\right)^c \tag{9.VIII.l}$$

$$\left.\begin{array}{rrr} \text{Mass}, M & b = & 1 \\ \text{Length}, L & a - 3b = & -1 \\ \text{time}, t & -c = & -2 \end{array}\right\} \Longrightarrow \pi_6 = \frac{\Delta P}{\rho\,N^2\,D^2} \tag{9.VIII.m}$$

The first dimensionless parameter π_1 represents the dimensionless flow rate. The second number represents the importance of the compressibility of the liquid in the pump. Some argue that this parameter is similar to Mach number (speed of disturbance to speed of sound. The third parameter is similar to Reynolds number since the combination $N\,D$ can be interpreted as velocity. The fourth number represents the production quality (mostly mode by some casting process[7]). The fifth dimensionless parameter is related to the ratio of the body forces to gravity forces. The last number represent the "effectiveness" of pump or can be viewed as dimensionless pressure obtained from the pump.

In practice, the roughness is similar to similar size pump and can be neglected. However, if completely different size of pumps are compared then this number must be considered. In cases where the compressibility of the liquid can be neglected or the pressure increase is relatively insignificant, the second dimensionless parameter can be neglected.

A pump is a device that intends to increase the pressure. The increase of the pressure involves energy inserted to to system. This energy is divided to a useful energy (pressure increase) and to overcome the losses in the system. These losses has several components which includes the friction in the system, change order of the flow and "ideal flow" loss. The most dominate loss in pump is loss of order, also know as turbulence (not covered yet this book.). If this physical phenomenon is accepted

[7]The modern production is made by die casting process. The reader is referred to "Fundamentals of die casting design," Genick Bar–Meir, Potto Project, 1999 to learn more.

than the resistance is neglected and the fourth parameter is removed. In that case the functional relationship can be written as

$$\frac{\Delta P}{N^2, D^2} = f\left(\frac{Q}{N\,D^3}\right) \tag{9.VIII.n}$$

End Solution

9.2.3.3 Mathematical Method: Constructing Dimensional Parameters

Advance material can be skipped

under construction please ignore for time being

In the progression of the development of the technique the new evolution is the mathematical method. It can be noticed that in the previous technique the same matrix was constructed with different vector solution (the right hand side of the equation). This fact is the source to improve the previous method. However, it has to be cautioned that this technique is overkill in most cases. Actually, this author is not aware for any case this technique has any advantage over the "building block" technique.

In the following hypothetical example demonstrates the reason for the reduction of variables. Assume that water is used to transport uniform grains of gold. The total amount grains of gold is to be determined per unit length. For this analysis it is assumed that grains of gold grains are uniformly distributed. The following parameters and their dimensions are considered:

Table -9.5. Units and Parameters of gold grains

Parameters	Units	Dimension	Remarks
grains amount	q	M/L	total grains per unit length
cross section area	A	L^2	pipe cross section
grains per volume	gr	$grains/L^3$	count of grain per V
grain weight	e	$M/grain$	count of grain per V

Notice that $grains$ and $grain$ are the same units for this discussion. Accordingly, the dimensional matrix can be constructed as

Table -9.6. gold grain dimensional matrix

	q	**A**	**gr**	**e**
M	1	0	0	1
L	1	2	3	0
grain	0	0	1	-1

In this case the total number variables are 4 and number basic units are 3. Thus, the total of one dimensional parameter.

End ignore section

End Advance material

9.2.4 Similarity and Similitude

One of dimensional analysis is the key point is the concept that the solution can be obtained by conducting experiments on similar but not identical systems. The analysis here suggests and demonstrates[8] that the solution is based on several dimensionless numbers. Hence, constructing experiments of the situation where the same dimensionless parameters obtains could, in theory, yield a solution to problem at hand. Thus, knowing what are dimensionless parameters should provide the knowledge of constructing the experiments.

In this section deals with these similarities which in the literature some refer as analogy or similitude. It is hard to obtain complete similarity. Hence, there is discussion how similar the model is to the prototype. It is common to differentiate between three kinds of similarities: geometric, kinetics, and dynamic. This characterization started because historical reasons and it, some times, has merit especially when applying Buckingham's method. In Nusselt's method this differentiation is less important.

Geometric Similarity

One of the logical part of dimensional analysis is how the experiences should be similar to actual body they are supposed to represent. This logical conclusion is an add–on and this author is not aware of any proof to this requirement based on Buckingham's methods. Ironically, this conclusion is based on Nusselt's method which calls for the same dimensionless boundary conditions. Again, Nusselt's method, sometimes or even often, requires similarity because the requirements to the boundary conditions. Here[9] this postulated idea is adapted.

[8]This statement is too strong. It has to be recognized that the results are as good as the guessing which in most cases is poor.

[9]Because this book intend to help students to pass their exams, this book present what most instructors required. It well established that this over–strict requirement and under Nusselt's method it can be overcome.

Under this idea the prototype area has to be square of the actual model or

$$\frac{A_p}{A_m} = \left(\frac{\ell_{1prototype}}{\ell_{1model}}\right)^2 = \left(\frac{\ell_{2p}}{\ell_{2m}}\right)^2 \tag{9.9}$$

where ℓ_1 and ℓ_2 are the typical dimensions in two different directions and subscript p refers to the prototype and m to the model. Under the same argument the volumes change with the cubes of lengths.

In some situations, the model faces inability to match two or more dimensionless parameters. In that case, the solution is to sacrifice the geometric similarity to minimize the undesirable effects. For example, river modeling requires to distort vertical scales to eliminate the influence of surface tension or bed roughness or sedimentation.

Kinematic Similarity

The perfect kinetics similarity is obtained when there are geometrical similarity and the motions of the fluid above the objects are the same. If this similarity is not possible, then the desire to achieve a motion "picture" which is characterized by ratios of corresponding velocities and accelerations is the same throughout the actual flow field. It is common in the literature, to discuss the situations there the model and prototype are similar but the velocities are different by a different scaling factor.

The geometrical similarity aside the shapes and counters of the object it also can requires surface roughness and erosion of surfaces of mobile surfaces or sedimentation of particles surface tensions. These impose demands require a minimum on the friction velocity. In some cases the minimum velocity can be $U_{min} = \sqrt{\tau_w/\rho}$. For example, there is no way achieve low Reynolds number with thin film flow.

Dynamics Similarity

The dynamic similarity has many confusing and conflicting definitions in the literature. Here this term refers to similarity of the forces. It follows, based on Newton's second law, that this requires that similarity in the accelerations and masses between the model and prototype. It was shown that the solution is a function of several typical dimensionless parameters. One of such dimensionless parameter is the Froude number. The solution for the model and the prototype are the same, since both cases have the same Froude number. Hence it can be written that

$$\left(\frac{U^2}{g\,\ell}\right)_m = \left(\frac{U^2}{g\,\ell}\right)_p \tag{9.10}$$

It can be noticed that $t \sim \ell/U$ thus equation (9.10) can be written as

$$\left(\frac{U}{g\,t}\right)_m = \left(\frac{U}{g\,t}\right)_p \tag{9.11}$$

and noticing that $a \propto U/t$

$$\left(\frac{a}{g}\right)_m = \left(\frac{a}{g}\right)_p \tag{9.12}$$

and $a \propto F/m$ and $m = \rho\,\ell^3$ hence $a = F/\rho\,\ell^3$. Substituting into equation (9.12) yields

$$\left(\frac{F}{\rho\,\ell^3}\right)_m = \left(\frac{F}{\rho\,\ell^3}\right)_p \Longrightarrow \frac{F_p}{F_m} = \frac{\left(\rho\,\ell^3\right)_p}{\left(\rho\,\ell^3\right)_m} \tag{9.13}$$

In this manipulation, it was shown that the ratio of the forces in the model and forces in the prototype is related to ratio of the dimensions and the density of the same systems. While in Buckingham's methods these hand waiving are not precise, the fact remains that there is a strong correlation between these forces. The above analysis was dealing with the forces related to gravity. A discussion about force related the viscous forces is similar and is presented for the completeness.

The Reynolds numbers is a common part of Navier–Stokes equations and if the solution of the prototype and for model to be same, the Reynolds numbers have to be same.

$$Re_m = Re_p \Longrightarrow \left(\frac{\rho\,U\,\ell}{\mu}\right)_m = \left(\frac{\rho\,U\,\ell}{\mu}\right)_p \tag{9.14}$$

Utilizing the relationship $U \propto \ell/t$ transforms equation (9.14) into

$$\left(\frac{\rho\,\ell^2}{\mu\,t}\right)_m = \left(\frac{\rho\,\ell^2}{\mu\,t}\right)_p \tag{9.15}$$

multiplying by the length on both side of the fraction by $\ell\,U$as

$$\left(\frac{\rho\,\ell^3\,U}{\mu\,t\,\ell\,U}\right)_m = \left(\frac{\rho\,\ell^3\,U}{\mu\,t\,\ell\,U}\right)_p \Longrightarrow \frac{\left(\rho\,\ell^3\,U/t\right)_m}{\left(\rho\,\ell^3\,U/t\right)_p} = \frac{(\mu\,\ell\,U)_m}{(\mu\,\ell\,U)_p} \tag{9.16}$$

Noticing that U/t is the acceleration and $\rho\,\ell$ is the mass thus the forces on the right hand side are proportional if the Re number are the same. In this analysis/discussion, it is assumed that a linear relationship exist. However, the Navier–Stokes equations are not linear and hence this assumption is excessive and this assumption can produce another source of inaccuracy.

While this explanation is a poor practice for the real world, it common to provide questions in exams and other tests on this issue. This section is provide to this purpose.

Example 9.9:

The liquid height rises in a tube due to the surface tension, σ is h. Assume that this height is a function of the body force (gravity, g), fluid density, ρ, radius, r, and the contact angle θ. Using Buckingham's theorem develop the relationship of the parameters. In experimental with a diameter 0.001 [m] and surface tension of 73 milli-Newtons/meter and contact angle of 75° a height is 0.01 [m] was obtained. In another situation, the surface tension is 146 milli-Newtons/meter, the diameter is 0.02 [m] and the contact angle and density remain the same. Estimate the height.

SOLUTION

It was given that the height is a function of several parameters such

$$h = f\left(\sigma, \rho, g, \theta, r\right) \tag{9.IX.a}$$

There are 6 parameters in the problem and the 3 basic parameters $[L, M, t]$. Thus the number of dimensionless groups is (6-3=3). In Buckingham's methods it is either that the angle isn't considered or the angle is dimensionless group by itself. Five parameters are left to form the next two dimensionless groups.

One technique that was suggested is the possibility to use three parameters which contain the basic parameters [M, L, t] and with them form a new group with each of the left over parameters. In this case, density, ρ for [M] and d for [L] and gravity, g for time [t]. For the surface tension, σ it becomes

$$\Bigg[\overbrace{M\,L^{-3}}^{\rho}\Bigg]^{a}\,[\overbrace{L}^{r}]^{b}\left[\overbrace{L\,t^{-2}}^{g}\right]^{c}\left[\overbrace{M\,t^{-2}}^{\sigma}\right]^{1} = M^0\,L^0\,t^0 \tag{9.IX.b}$$

Equation (9.IX.b) leads to three equations which are

$$\begin{aligned} \text{Mass}, M \qquad a+1 &= 0 \\ \text{Length}, L \quad -3a+b+c &= 0 \\ \text{time}, t \qquad -2c-2 &= 0 \end{aligned} \tag{9.IX.c}$$

the solution is $a = -1 \quad b = -2 \quad c = -1$ Thus the dimensionless group is $\dfrac{\sigma}{\rho\,r^2\,g}$. The third group obtained under the same procedure to be h/r.

In the second part the calculations for the estimated of height based on the new ratios. From the above analysis the functional dependency can be written as

$$\frac{h}{d} = f\left(\frac{\sigma}{\rho\,r^5\,g}\,,\,\theta\right) \tag{9.IX.d}$$

which leads to the same angle and the same dimensional number. Hence,

$$\frac{h_1}{d_1} = \frac{h_2}{d_2} = f\left(\frac{\sigma}{\rho\,r^2\,g}\,,\,\theta\right) \tag{9.IX.e}$$

Since the dimensionless parameters remain the same, the ratio of height and radius must be remain the same. Hence,

$$h_2 = \frac{h_1\,d_2}{d_1} = \frac{0.01 \times 0.002}{0.001} = 0.002 \tag{9.IX.f}$$

End Solution

9.3 Nusselt's Technique

The Nusselt's method is a bit more labor intensive, in that the governing equations with the boundary and initial conditions are used to determine the dimensionless parameters. In this method, the boundary conditions together with the governing equations are taken into account as opposed to Buckingham's method. A common mistake is to ignore the boundary conditions or initial conditions. The parameters that results from this process are the dimensional parameters which control the problems. An example comparing the Buckingham's method with Nusselt's method is presented.

In this method, the governing equations, initial condition and boundary conditions are normalized resulting in a creation of dimensionless parameters which govern the solution. It is recommended, when the reader is out in the real world to simply abandon Buckingham's method all together. This point can be illustrated by example of flow over inclined plane. For comparison reasons Buckingham's method presented and later the results are compared with the results from Nusselt's method.

Example 9.10:

Utilize the Buckingham's method to analyze a two dimensional flow in incline plane. Assume that the flow infinitely long and thus flow can be analyzed per width which is a function of several parameters. The potential parameters are the angle of inclination, θ*, liquid viscosity,* ν*, gravity,* g*, the height of the liquid,* h*, the density,* ρ*, and liquid velocity,* U*. Assume that the flow is not affected by the surface tension (liquid),* σ*. You furthermore are to assume that the flow is stable. Develop the relationship between the flow to the other parameters.*

Solution

Under the assumptions in the example presentation leads to following

$$\dot{m} = f\left(\theta, \nu, g, \rho, U\right) \tag{9.17}$$

The number of basic units is three while the number of the parameters is six thus the difference is $6 - 3 = 3$. Those groups (or the work on the groups creation) further can be reduced the because angle θ is dimensionless. The units of parameters can be obtained in Table 9.3 and summarized in the following table.

Table -9.7. Units of the Pendulum Parameters

Parameter	Units	Parameter	Units	Parameter	Units
ν	L^2t^{-1}	g	$L^1t{-2}$	U	L^1t^{-1}
$\dot{m}$	$Mt^{-1}L^{-1}$	θ	none	ρ	ML^3

The basic units are chosen as for the time, U, for the mass, ρ, and for the length g. Utilizing the building blocks technique provides

$$\overbrace{\frac{M}{t\,L}}^{\dot{m}} = \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^a \left(\overbrace{\frac{L}{t^2}}^{g}\right)^b \left(\overbrace{\frac{L}{t}}^{U}\right)^c \tag{9.X.a}$$

The equations obtained from equation (9.X.a) are

$$\left.\begin{array}{rrr} \text{Mass}, M & a = & 1 \\ \text{Length}, L & -3a + b + c = & -1 \\ \text{time}, t & -2b - c = & -1 \end{array}\right\} \Longrightarrow \pi_1 = \frac{\dot{m}\,g}{\rho\,U^3} \tag{9.X.b}$$

$$\overbrace{\frac{L^2}{t}}^{\nu} = \left(\overbrace{\frac{M}{L^3}}^{\rho}\right)^a \left(\overbrace{\frac{L}{t^2}}^{g}\right)^b \left(\overbrace{\frac{L}{t}}^{U}\right)^c \tag{9.X.c}$$

The equations obtained from equation (9.X.a) are

$$\left.\begin{array}{rrr} \text{Mass}, M & a = & 0 \\ \text{Length}, L & -3a + b + c = & 2 \\ \text{time}, t & -2b - c = & -1 \end{array}\right\} \Longrightarrow \pi_2 = \frac{\nu\,g}{U^3} \tag{9.X.d}$$

Thus governing equation and adding the angle can be written as

$$0 = f\left(\frac{\dot{m}\,g}{\rho\,U^3}, \frac{\nu\,g}{U^3}, \theta\right) \tag{9.X.e}$$

The conclusion from this analysis are that the number of controlling parameters totaled in three and that the initial conditions and boundaries are irrelevant.

End Solution

A small note, it is well established that the combination of angle gravity or effective body force is significant to the results. Hence, this analysis misses, at the very least, the issue of the combination of the angle gravity. Nusselt's analysis requires that the governing equations along with the boundary and initial conditions to be written. While the analytical solution for this situation exist, the parameters that effect the problem are the focus of this discussion.

In Chapter 8, the Navier–Stokes equations were developed. These equations along with the energy, mass or the chemical species of the system, and second laws governed almost all cases in thermo–fluid mechanics. This author is not aware of a compelling reason that this fact[10] should be used in this chapter. The two dimensional NS equation can obtained from equation (8.VIII.a) as

$$\begin{aligned}\rho\left(\frac{\partial U_x}{\partial t}+\right. & \left. U_x\frac{\partial U_x}{\partial x}+U_y\frac{\partial U_x}{\partial y}+U_z\frac{\partial U_x}{\partial z}\right)= \\ & -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 U_x}{\partial x^2}+\frac{\partial^2 U_x}{\partial y^2}+\frac{\partial^2 U_x}{\partial z^2}\right)+\rho g\sin\theta\end{aligned} \tag{9.18}$$

and

$$\begin{aligned}\rho\left(\frac{\partial U_y}{\partial t}+\right. & \left. U_x\frac{\partial U_y}{\partial x}+U_y\frac{\partial U_y}{\partial y}+U_z\frac{\partial U_y}{\partial z}\right)= \\ & -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 U_y}{\partial x^2}+\frac{\partial^2 U_y}{\partial y^2}+\frac{\partial^2 U_y}{\partial z^2}\right)+\rho g\sin\theta\end{aligned} \tag{9.19}$$

With boundary conditions

$$\begin{aligned}& U_x(y=0)=U_{0x}f(x) \\ & \frac{\partial U_x}{\partial x}(y=h)=\tau_0 f(x)\end{aligned} \tag{9.20}$$

The value U_0x and τ_0 are the characteristic and maximum values of the velocity or the shear stress, respectively. and the initial condition of

$$U_x(x=0)=U_{0y}\,f(y) \tag{9.21}$$

where U_{0y} is characteristic initial velocity.

These sets of equations (9.18)–(9.21) need to be converted to dimensionless equations. It can be noticed that the boundary and initial conditions are provided in a special form were the representative velocity multiply a function. Any function can be presented by this form.

In the process of transforming the equations into a dimensionless form associated with some intelligent guess work. However, no assumption is made or required about whether or not the velocity, in the y direction. The only exception is that the y component of the velocity vanished on the boundary. No assumption is required about the acceleration or the pressure gradient etc.

The boundary conditions have typical velocities which can be used. The velocity is selected according to the situation or the needed velocity. For example, if the effect of the initial condition is under investigation than the characteristic of that velocity should be used. Otherwise the velocity at the bottom should be used. In that case, the

[10] In economics and several other areas, there are no governing equations established for the field nor there is necessarily concept of conservation of something. However, writing the governing equations will yield dimensionless parameters as good as the initial guess.

boundary conditions are

$$\frac{U_x(y=0)}{U_{0x}} = f(x)$$
$$\mu \frac{\partial U_x}{\partial x}(y=h) = \tau_0\, g(x) \tag{9.22}$$

Now it is very convenient to define several new variables:

$$\overline{U} = \frac{U_x(\overline{x})}{U_{0x}}$$
$$where:$$
$$\overline{x} = \frac{x}{h} \qquad \overline{y} = \frac{y}{h} \tag{9.23}$$

The length h is chosen as the characteristic length since no other length is provided. It can be noticed that because the units consistency, the characteristic length can be used for "normalization" (see Example 9.11). Using these definitions the boundary and initial conditions becomes

$$\frac{\overline{U_x}(\overline{y}=0)}{U_{0x}} = f^{'}(\overline{x})$$
$$\frac{h\,\mu}{U_{0x}}\frac{\partial \overline{U_x}}{\partial \overline{x}}(\overline{y}=1) = \tau_0\, g^{'}(\overline{x}) \tag{9.24}$$

It commonly suggested to arrange the second part of equation (9.24) as

$$\frac{\partial \overline{U_x}}{\partial \overline{x}}(\overline{y}=1) = \frac{\tau_0\, U_{0x}}{h\,\mu}\, g^{'}(\overline{x}) \tag{9.25}$$

Where new dimensionless parameter, the shear stress number is defined as

$$\overline{\tau_0} = \frac{\tau_0\, U_{0x}}{h\,\mu} \tag{9.26}$$

With the new definition equation (9.25) transformed into

$$\frac{\partial \overline{U_x}}{\partial \overline{x}}(\overline{y}=1) = \overline{\tau_0}\, g^{'}(\overline{x}) \tag{9.27}$$

Example 9.11:

Non–dimensionalize the following boundary condition. What are the units of the coefficient in front of the variables, x. What are relationship of the typical velocity, U_0 to U_{max}?

$$U_x(y=h) = U_0\left(a\,x^2 + b\,\exp(x)\right) \tag{9.XI.a}$$

Solution

The coefficients a and b multiply different terms and therefore must have different units. The results must be unitless thus a

$$L^0 = a \overbrace{L^2}^{x^2} \Longrightarrow a = \left[\frac{1}{L^2}\right] \tag{9.XI.b}$$

From equation (9.XI.b) it clear the conversion of the first term is $U_x = a\,h^2\overline{x}$. The exponent appears a bit more complicated as

$$L^0 = b\,\exp\left(h\,\frac{x}{h}\right) = b\,\exp\left(h\right)\,\exp\left(\frac{x}{h}\right) = b\,\exp\left(h\right)\,\exp\left(\overline{x}\right) \tag{9.XI.c}$$

Hence defining

$$\overline{b} = \frac{1}{\exp h} \tag{9.XI.d}$$

With the new coefficients for both terms and noticing that $y = h \longrightarrow \overline{y} = 1$ now can be written as

$$\frac{U_x(\overline{y}=1)}{U_0} = \overbrace{a\,h^2}^{\overline{a}}\;x^2 + \overbrace{b\,\exp\left(h\right)}^{\overline{b}}\,\exp\left(\overline{x}\right) = \overline{a}\,\overline{x}^2 + \overline{b}\exp\overline{x} \tag{9.XI.e}$$

Where $\overline{a}$ and $\overline{b}$ are the transformed coefficients in the dimensionless presentation.

End Solution

After the boundary conditions the initial condition can undergo the non–dimensional process. The initial condition (9.21) utilizing the previous definitions transformed into

$$\frac{U_x(\overline{x}=0)}{U_{0x}} = \frac{U_{0y}}{U_{0x}}\,f(\overline{y}) \tag{9.28}$$

Notice the new dimensionless group of the velocity ratio as results of the boundary condition. This dimensionless number was and cannot be obtained using the Buckingham's technique. The physical significance of this number is an indication to the "penetration" of the initial (condition) velocity.

The main part of the analysis if conversion of the governing equation into a dimensionless form uses previous definition with additional definitions. The dimensionless time is defined as $\overline{t} = t\,U_{0x}/h$. This definition based on the characteristic time of h/U_{0x}. Thus, the derivative with respect to time is

$$\frac{\partial U_x}{\partial t} = \frac{\partial\,\overbrace{\overline{U_x}}^{\frac{U_x}{U_{0x}}}\,U_{0x}}{\partial\,\underbrace{\overline{t}}_{\frac{t\,U_{0x}}{h}}\,\frac{h}{U_{0x}}} = \frac{{U_{0x}}^2}{h}\,\frac{\partial\overline{U_x}}{\partial\overline{t}} \tag{9.29}$$

Notice that the coefficient has units of acceleration. The second term

$$U_x \frac{\partial U_x}{\partial x} = \overbrace{\overline{U_x}}^{\frac{U_x}{U_{0x}}} U_{0x} \frac{\partial \overbrace{\overline{U_x}}^{\frac{U_x}{U_{0x}}} U_{0x}}{\partial \underbrace{\overline{x}}_{\frac{x}{h}} h} = \frac{{U_{0x}}^2}{h} \overline{U_x} \frac{\partial \overline{U_x}}{\partial \overline{x}} \tag{9.30}$$

The pressure is normalized by the same initial pressure or the static pressure as $(P - P_\infty) / (P_0 - P_\infty)$ and hence

$$\frac{\partial P}{\partial x} = \frac{\partial \overbrace{\overline{P}}^{\frac{P-P_\infty}{P_0-P_\infty}}}{\partial \overline{x} h} (P_0 - P_\infty) = \frac{(P_0 - P_\infty)}{h} \frac{\partial \overline{P}}{\partial \overline{x}} \tag{9.31}$$

The second derivative of velocity looks like

$$\frac{\partial^2 U_x}{\partial x^2} = \frac{\partial}{\partial (\overline{x} h)} \frac{\partial \left(\overline{U_x} U_{0x}\right)}{\partial (\overline{x} h)} = \frac{U_{0x}}{h^2} \frac{\partial^2 \overline{U_x}}{\partial \overline{x}^2} \tag{9.32}$$

The last term is the gravity g which is left for the later stage. Substituting all terms and dividing by density, ρ result in

$$\begin{gathered}\frac{{U_{0x}}^2}{h} \left(\frac{\partial \overline{U_x}}{\partial \overline{t}} + \quad \overline{U_x} \frac{\partial \overline{U_x}}{\partial \overline{x}} + \overline{U_y} \frac{\partial \overline{U_x}}{\partial \overline{y}} + \overline{U_z} \frac{\partial \overline{U_x}}{\partial \overline{z}} \right) = \\ - \frac{P_0 - P_\infty}{h\,\rho} \frac{\partial \overline{P}}{\partial \overline{x}} + \frac{U_{0x} \mu}{h^2\,\rho} \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right) + \frac{\not{\rho} g}{\not{\rho}} \sin\theta\end{gathered} \tag{9.33}$$

Dividing equation (9.33) by ${U_{0x}}^2/h$ yields

$$\begin{gathered}\left(\frac{\partial \overline{U_x}}{\partial \overline{t}} + \quad \overline{U_x} \frac{\partial \overline{U_x}}{\partial \overline{x}} + \overline{U_y} \frac{\partial \overline{U_x}}{\partial \overline{y}} + \overline{U_z} \frac{\partial \overline{U_x}}{\partial \overline{z}} \right) = \\ - \frac{P_0 - P_\infty}{{U_{0x}}^2\,\rho} \frac{\partial \overline{P}}{\partial \overline{x}} + \frac{\mu}{U_{0x}\,h\,\rho} \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right) + \frac{g\,h}{{U_{0x}}^2} \sin\theta\end{gathered} \tag{9.34}$$

Defining "initial" dimensionless parameters as

$$Re = \frac{U_{0x}\,h\,\rho}{\mu} \qquad Fr = \frac{U_{0x}}{\sqrt{g\,h}} \qquad Eu = \frac{P_0 - P_\infty}{{U_{0x}}^2\,\rho} \tag{9.35}$$

Substituting definition of equation (9.35) into equation (9.36) yields

$$\begin{gathered}\left(\frac{\partial \overline{U_x}}{\partial \overline{t}} + \quad \overline{U_x} \frac{\partial \overline{U_x}}{\partial \overline{x}} + \overline{U_y} \frac{\partial \overline{U_x}}{\partial \overline{y}} + \overline{U_z} \frac{\partial \overline{U_x}}{\partial \overline{z}} \right) = \\ -Eu \frac{\partial \overline{P}}{\partial \overline{x}} + \frac{1}{Re} \left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2} \right) + \frac{1}{Fr^2} \sin\theta\end{gathered} \tag{9.36}$$

Equation (9.36) show one common possibility of a dimensionless presentation of governing equation. The significance of the large and small value of the dimensionless parameters will be discuss later in the book. Without actually solving the problem, Nusselt's method provides several more parameters that were not obtained by the block method. The solution of the governing equation is a function of all the parameters present in that equation and boundaries condition as well the initial condition. Thus, the solution is

$$U_x = f\left(\overline{x}, \overline{y}, Eu, Re, Fr, \theta, \overline{\tau_0}, f_u, f_\tau, \frac{U_{0y}}{U_{0x}}\right) \tag{9.37}$$

The values of $\overline{x}$, $\overline{y}$ depend on h and hence the value of h is an important parameter.

It can be noticed with Buckingham's method, the number of parameters obtained was only three (3) while Nusselt's method yields 12 dimensionless parameters. This is a very significant difference between the two methods. In fact, there are numerous examples in the literature that showing people doing experiments based on Buckingham's methods. In these experiments, major parameters are ignored rendering these experiments useless in many cases and deceiving.

Common Transformations

Fluid mechanics in particular and Thermo–Fluid field in general have several common transformations that appear in boundary conditions, initial conditions and equations[11] . It recognized that not all the possibilities can presented in the example shown above. Several common boundary conditions which were not discussed in the above example are presented below. As an initial matter, the results of the non dimensional transformation depends on the selection of what and how is nondimensionalization carried. This section of these parameters depends on what is investigated. Thus, one of the general nondimensionalization of the Navier–Stokes and energy equations will be discussed at end of this chapter.

Boundary conditions are divided into several categories such as a given value to the function[12], given derivative (Neumann b.c.), mixed condition, and complex conditions. The first and second categories were discussed to some degree earlier and will be expanded later. The third and fourth categories were not discussed previously. The non–dimensionalization of the boundary conditions of the first category requires finding and diving the boundary conditions by a typical or a characteristic value. The second category involves the nondimensionalization of the derivative. In general, this process involve dividing the function by a typical value and the same for length variable (e.g. x) as

$$\frac{\partial U}{\partial x} = \frac{\ell}{U_0} \frac{\partial \left(\frac{U}{U_0}\right)}{\partial \left(\frac{x}{\ell}\right)} = \frac{\ell}{U_0} \frac{\partial \overline{U}}{\partial \overline{x}} \tag{9.38}$$

[11]Many of these tricks spread in many places and fields. This author is not aware of a collection of this kind of transforms.

[12]The mathematicians like to call Dirichlet conditions

In the Thermo–Fluid field and others, the governing equation can be of higher order than second order[13]. It can be noticed that the degree of the derivative boundary condition cannot exceed the derivative degree of the governing equation (e.g. second order equation has at most the second order differential boundary condition.). In general "nth" order differential equation leads to

$$\frac{\partial^n U}{\partial x^n} = \frac{U_0}{\ell^n} \frac{\partial^n \left(\frac{U}{U_0}\right)}{\partial \left(\frac{x}{\ell}\right)^n} = \frac{U_0}{\ell^n} \frac{\partial^n \overline{U}}{\partial \overline{x}^n} \tag{9.39}$$

The third kind of boundary condition is the mix condition. This category includes combination of the function with its derivative. For example a typical heat balance at liquid solid interface reads

$$h(T_0 - T) = -k \frac{\partial T}{\partial x} \tag{9.40}$$

This kind of boundary condition, since derivative of constant is zero, translated to

$$h \cancel{(T_0 - T_{max})} \left(\frac{T_0 - T}{T_0 - T_{max}} \right) = - \frac{k \cancel{(T_0 - T_{max})}}{\ell} \frac{-\partial \left(\dfrac{T - T_0}{T_0 - T_{max}} \right)}{\partial \left(\dfrac{x}{\ell} \right)} \tag{9.41}$$

or

$$\left(\frac{T_0 - T}{T_0 - T_{max}} \right) = \frac{k}{h\,\ell} \frac{\partial \left(\dfrac{T - T_0}{T_0 - T_{max}} \right)}{\partial \left(\dfrac{x}{\ell} \right)} \Longrightarrow \Theta = \frac{1}{Nu} \frac{\partial \Theta}{\partial \overline{x}} \tag{9.42}$$

Where Nusselt Number and the dimensionless temperature are defined as

$$Nu = \frac{h\,\ell}{k} \qquad \Theta = \frac{T - T_0}{T_0 - T_{max}} \tag{9.43}$$

and T_{max} is the maximum or reference temperature of the system.

The last category is dealing with some non–linear conditions of the function with its derivative. For example,

$$\Delta P \approx \sigma \left(\frac{1}{r_1} + \frac{1}{r_2} \right) = \frac{\sigma}{r_1} \frac{r_1 + r_2}{r_2} \tag{9.44}$$

Where r_1 and r_2 are the typical principal radii of the free surface curvature, and, σ, is the surface tension between the gas (or liquid) and the other phase. The surface geometry (or the radii) is determined by several factors which include the liquid movement

[13] This author aware of fifth order partial differential governing equations in some cases. Thus, the highest derivative can be fifth order derivative.

instabilities etc chapters of the problem at hand. This boundary condition (9.45) can be rearranged to be

$$\frac{\Delta P\, r_1}{\sigma} \approx \frac{r_1 + r_2}{r_2} \Longrightarrow Av \approx \frac{r_1 + r_2}{r_2} \tag{9.45}$$

Where Av is Avi number . The Avi number represents the geometrical characteristics combined with the material properties. The boundary condition (9.45) can be transferred into

$$\frac{\Delta P\, r_1}{\sigma} = Av \tag{9.46}$$

Where ΔP is the pressure difference between the two phases (normally between the liquid and gas phase).

One of advantage of Nusselt's method is the Object–Oriented nature which allows one to add additional dimensionless parameters for addition "degree of freedom." It is common assumption, to initially assume, that liquid is incompressible. If greater accuracy is needed than this assumption is removed. In that case, a new dimensionless parameters is introduced as the ratio of the density to a reference density as

$$\overline{\rho} = \frac{\rho}{\rho_0} \tag{9.47}$$

In case of ideal gas model with isentropic flow this assumption becomes

$$\bar{\rho} = \frac{\rho}{\rho_0} = \left(\frac{P_0}{P}\right)^{\frac{1}{n}} \tag{9.48}$$

The power n depends on the gas properties.

Characteristics Values

Normally, the characteristics values are determined by physical values e.g. The diameter of cylinder as a typical length. There are several situations where the characteristic length, velocity, for example, are determined by the physical properties of the fluid(s). The characteristic velocity can determined from $U_0 = \sqrt{2P_0/\rho}$. The characteristic length can be determined from ratio of $\ell = \Delta P/\sigma$.

Example 9.12:

One idea of renewable energy is to use and to utilize the high concentration of of brine water such as in the Salt Lake and the Salt Sea (in Israel). This process requires analysis the mass transfer process. The governing equation is non–linear and this example provides opportunity to study nondimensionalizing of this kind of equation. The conversion of the species yields a governing nonlinear equation[14] for such process is

$$U_0 \frac{\partial C_A}{\partial x} = \frac{\partial}{\partial y} \frac{D_{AB}}{(1 - X_A)} \frac{\partial C_A}{\partial y} \tag{9.XII.a}$$

Where the concentration, C_A is defended as the molar density i.e. the number of moles per volume. The molar fraction, X_A is defined as the molar fraction of species A divide by the total amount of material (in moles). The diffusivity coefficient, D_{AB} is defined as penetration of species A into the material. What are the units of the diffusivity coefficient? The boundary conditions of this partial differential equation are given by

$$\frac{\partial C_A}{\partial y}\left(y=\infty\right) = 0 \tag{9.XII.b}$$

$$C_A(y=0) = C_e \tag{9.XII.c}$$

Where C_e is the equilibrium concentration. The initial condition is

$$C_A(x=0) = C_0 \tag{9.XII.d}$$

Select dimensionless parameters so that the governing equation and boundary and initial condition can be presented in a dimensionless form. There is no need to discuss the physical significance of the problem.

SOLUTION

This governing equation requires to work with dimension associated with mass transfer and chemical reactions, the "mole." However, the units should not cause confusion or fear since it appear on both sides of the governing equation. Hence, this unit will be canceled. Now the units are compared to make sure that diffusion coefficient is kept the units on both sides the same. From units point of view, equation (9.XII.a) can be written (when the concentration is simply ignored) as

$$\overbrace{\frac{L}{t}}^{U}\overbrace{\frac{\not{C}}{L}}^{\frac{\partial C}{\partial x}} = \overbrace{\frac{1}{L}}^{\frac{\partial}{\partial y}}\overbrace{\frac{D_{AB}}{1}}^{\frac{D_{AB}}{(1-X)}}\overbrace{\frac{\not{C}}{L}}^{\frac{\partial C}{\partial y}} \tag{9.XII.e}$$

It can be noticed that X is unitless parameter because two same quantities are divided.

$$\frac{1}{t} = \frac{1}{L^2}D_{AB} \Longrightarrow D_{AB} = \frac{L^2}{t} \tag{9.XII.f}$$

Hence the units of diffusion coefficient are typically given by $\left[m^2/sec\right]$ (it also can be observed that based on Fick's laws of diffusion it has the same units).

The potential of possibilities of dimensionless parameter is large. Typically, dimensionless parameters are presented as ratio of two quantities. In addition to that, in heat and mass transfer (also in pressure driven flow etc.) the relative or reference to certain point has to accounted for. The boundary and initial conditions here provides

[14]More information how this equation was derived can be found in Bar–Meir (Meyerson), Genick "Hygroscopic absorption to falling films: The effects of the concentration level" M.S. Thesis Tel-Aviv Univ. (Israel). Dept. of Fluid Mechanics and Heat Transfer 12/1991.

the potential of the "driving force" for the mass flow or mass transfer. Hence, the potential definition is

$$\Phi = \frac{C_A - C_0}{C_e - C_0} \tag{9.XII.g}$$

With almost "standard" transformation

$$\overline{x} = \frac{x}{\ell} \qquad \overline{y} = \frac{y}{\ell} \tag{9.XII.h}$$

Hence the derivative of Φ with respect to time is

$$\frac{\partial \Phi}{\partial \overline{x}} = \frac{\partial \dfrac{C_A - C_0}{C_e - C_0}}{\partial \dfrac{x}{\ell}} = \frac{\ell}{C_e - C_0} \frac{\partial \left(C_A - \cancelto{0}{C_0} \right)}{\partial x} = \frac{\ell}{C_e - C_0} \frac{\partial C_A}{\partial x} \tag{9.XII.i}$$

In general a derivative with respect to $\overline{x}$ or $\overline{y}$ leave yields multiplication of ℓ. Hence, equation (9.XII.a) transformed into

$$\begin{aligned} U_0 \frac{\cancel{(C_e - C_0)}}{\ell} \frac{\partial \Phi}{\partial \overline{x}} &= \frac{1}{\ell} \frac{\partial}{\partial \overline{y}} \frac{D_{AB}}{(1 - X_A)} \frac{\cancel{(C_e - C_0)}}{\ell} \frac{\partial \Phi}{\partial \overline{y}} \\ \rightsquigarrow \frac{U_0}{\ell} \frac{\partial \Phi}{\partial \overline{x}} &= \frac{1}{\ell^2} \frac{\partial}{\partial \overline{y}} \frac{D_{AB}}{(1 - X_A)} \frac{\partial \Phi}{\partial \overline{y}} \end{aligned} \tag{9.XII.j}$$

Equation (9.XII.j) like non–dimensionalized and proper version. However, the term X_A, while is dimensionless, is not proper. Yet, X_A is a function of Φ because it contains C_A. Hence, this term, X_A has to be converted or presented by Φ. Using the definition of X_A it can be written as

$$X_A = \frac{C_A}{C} = (C_e - C_0) \frac{C_A - C_0}{C_e - C_0} \frac{1}{C} \tag{9.XII.k}$$

Thus the transformation in equation (9.XII.l) another unexpected dimensionless parameter as

$$X_A = \Phi \frac{C_e - C_0}{C} \tag{9.XII.l}$$

Thus number, $\frac{C_e - C_0}{C}$ was not expected and it represent ratio of the driving force to the height of the concentration which was not possible to attend by Buckingham's method.

End Solution

9.4 Summary of Dimensionless Numbers

This section summarizes all the major dimensionless parameters which are commonly used in the fluid mechanics field.

Table -9.8. Common Dimensionless Parameters of Thermo–Fluid in the Field

Name	Symbol	Equation	Interpretation	Application
Archimede Number	Ar	$\dfrac{g\,\ell^3\rho_f(\rho-\rho_f)}{\mu^2}$	$\dfrac{\text{buoyancy forces}}{\text{viscous force}}$	in nature and force convection
Atwood Number	A	$\dfrac{(\rho_a-\rho_b)}{\rho_a+\rho_b}$	$\dfrac{\text{buoyancy forces}}{\text{"penetration" force}}$	in stability of liquid layer a over b RayleighTaylor instability etc.
Bond Number	Bo	$\dfrac{\rho\,g\,\ell^2}{\sigma}$	$\dfrac{\text{gravity forces}}{\text{surface tension force}}$	in open channel flow, thin film flow
Brinkman Number	Br	$\dfrac{\mu U^2}{k\,\Delta T}$	$\dfrac{\text{heat dissipation}}{\text{heat conduction}}$	during dissipation problems
Capillary Number	Ca	$\dfrac{\mu\,U}{\sigma}$	$\dfrac{\text{viscous force}}{\text{surface tension force}}$	For small Re and surface tension involve problem
Cauchy Number	Cau	$\dfrac{\rho\,U^2}{E}$	$\dfrac{\text{inertia force}}{\text{elastic force}}$	For large Re and surface tension involve problem
Cavitation Number	$\boldsymbol{\sigma}$	$\dfrac{P_l-P_v}{\frac{1}{2}\rho U^2}$	$\dfrac{\text{pressure difference}}{\text{inertia energy}}$	pressure difference to vapor pressure to the potential of phase change (mostly to gas)
Courant Number	Co	$\dfrac{\Delta t\,U}{\Delta x}$	$\dfrac{\text{wave distance}}{\text{Typical Distance}}$	A requirement in numerical schematic to achieve stability)
Dean Number	D	$\dfrac{Re}{\sqrt{R/h}}$	$\dfrac{\text{inertia forces}}{\text{viscous deviation forces}}$	related to radius of channel with width h stability
Deborah Number[15]	De	$\dfrac{t_c}{t_p}$	$\dfrac{\text{stress relaxation time}}{\text{observation time}}$	the ratio of the fluidity of material primary used in rheology
Drag Coefficient	C_D	$\dfrac{D}{\frac{1}{2}\,\rho\,U^2\,A}$	$\dfrac{\text{drag force}}{\text{inertia effects}}$	Aerodynamics, hydrodynamics, note this coefficient has many definitions
Eckert Number	Ec	$\dfrac{U^2}{C_p\,\Delta T}$	$\dfrac{\text{inertia effects}}{\text{thermal effects}}$	during dissipation problems

Continued on next page

Table -9.8. Common Dimensionless Parameters of Fluid Mechanics (continue)

Standard System				
Name	Symbol	Equation	Interpretation	Application
Ekman Number	Ek	$\dfrac{\nu}{2\ell^2\,\omega}$	$\dfrac{\text{viscous forces}}{\text{Coriolis forces}}$	geophysical flow like atmospheric flow
Euler Number	Ec	$\dfrac{P_0 - P_\infty}{C_p\,\Delta T}$	$\dfrac{\text{pressure potential effects}}{\text{inertia effects}}$	potential of resistance problems
Froude Number	Fr	$\dfrac{U}{\sqrt{g\,\ell}}$	$\dfrac{\text{inertia effects}}{\text{gravitational effects}}$	open channel flow and two phase flow
Galileo Number	Ga	$\dfrac{\rho\,g\,\ell^3}{\mu^2}$	$\dfrac{\text{gravitational effects}}{\text{viscous effects}}$	open channel flow and two phase flow
Grashof Number	Gr	$\dfrac{\beta\,\Delta T\,g\,\ell^3\,\rho^2}{\mu^2}$	$\dfrac{\text{buoyancy effects}}{\text{viscous effects}}$	natural convection
Knudsen Number	Kn	$\dfrac{\lambda}{\ell}$	$\dfrac{\text{LMFP}}{\text{characteristic length}}$	length of mean free path, LMFP, to characteristic length
Laplace Constant	La	$\sqrt{\dfrac{2\,\sigma}{g(\rho_1 - \rho_2)}}$	$\dfrac{\text{surface force}}{\text{gravity effects}}$	liquid raise, surface tension problem, also ref:Capillary constant
Lift Coefficient	C_L	$\dfrac{L}{\frac{1}{2}\,\rho\,U^2\,A}$	$\dfrac{\text{lift force}}{\text{inertia effects}}$	Aerodynamics, hydrodynamics, note this coefficient has many definitions
Mach Number	M	$\dfrac{U}{c}$	$\dfrac{\text{velocity}}{\text{sound speed}}$	compressibility and propagation of disturbances
Marangoni Number	Ma	$-\dfrac{d\sigma}{dT}\,\dfrac{\ell\,\Delta T}{\nu\alpha}$	$\dfrac{\text{“thermal” surface tension}}{\text{viscous force}}$	surface tension caused by thermal gradient
Morton Number	Mo	$\dfrac{g\mu_c^4\,\Delta\rho}{\rho_c^2\sigma^3}$	$\dfrac{\text{viscous force}}{\text{surface tension force}}$	bubble and drop flow
Ozer Number	Oz	$\dfrac{\frac{{C_D}^2\,P_{max}}{\rho}}{\left(\frac{Q_{max}}{A}\right)^2}$	$\dfrac{\text{“maximum” supply}}{\text{“maximum” demand}}$	supply and demand analysis such pump & pipe system, economy
				Continued on next page

Table -9.8. Common Dimensionless Parameters of Fluid Mechanics (continue)

Standard System				
Name	Symbol	Equation	Interpretation	Application
Prandtl Number	Pr	$\dfrac{\nu}{\alpha}$	$\dfrac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}$	Prandtl is fluid property important in flow due to thermal forces
Reynolds Number	Re	$\dfrac{\rho\, U\, \ell}{\mu}$	$\dfrac{\text{inertia forces}}{\text{viscous forces}}$	In most fluid mechanics issues
Rossby Number	Ro	$\dfrac{U}{\omega\, \ell_0}$	$\dfrac{\text{inertia forces}}{\text{Coriolis forces}}$	In rotating fluids
Shear Number	Sn	$\dfrac{\tau_c\, \ell_c}{\mu_c\, U_c}$	$\dfrac{\text{actual shear}}{\text{"potential" shear}}$	shear flow
Stokes Number	Stk	$\dfrac{t_p}{t_K}$	$\dfrac{\text{particle relaxation time}}{\text{Kolmogorov time}}$	In aerosol flow dealing with penetration of particles
Strouhal Number	St	$\dfrac{\omega\, \ell}{U}$	$\dfrac{\text{"unsteady" effects}}{\text{inertia effect}}$	The effects of natural or forced frequency in all the field that is how much the "unsteadiness" of the flow is
Taylor Number	Ta	$\dfrac{\rho^2\, {\omega_i}^2\, \ell^4}{\mu^4}$	$\dfrac{\text{centrifugal forces}}{\text{viscous forces}}$	Stability of rotating cylinders Notice ℓ has special definition
Weber Number	We	$\dfrac{\rho\, U^2\, \ell}{\sigma}$	$\dfrac{\text{inertia force}}{\text{surface tension force}}$	For large Re and surface tension involve problem

The dimensional parameters that were used in the construction of the dimensionless parameters in Table 9.8 are the characteristics of the system. Therefore there are several definition of Reynolds number. In fact, in the study of the physical situations often people refers to local Re number and the global Re number. Keeping this point in mind, there several typical dimensions which need to be mentioned. The typical body force is the gravity g which has a direction to center of Earth. The elasticity E in case of liquid phase is B_T, in case of solid phase is Young modulus. The typical length is denoted as ℓ and in many cases it is referred to as the diameter or the radius. The

[15]This number is named by Reiner, M. (1964), "The Deborah Number", Physics Today 17 (1): 62, doi:10.1063/1.3051374. Reiner, a civil engineer who is considered the father of Rheology, named this parameter because theological reasons perhaps since he was living in Israel.

density, ρ is referred to the characteristic density or density at infinity. The area, A in drag and lift coefficients is referred normally to projected area.

The frequency ω or f is referred to as the "unsteadiness" of the system. Generally, the periodic effect is enforced by the boundary conditions or the initial conditions. In other situations, the physics itself instores or forces periodic instability. For example, flow around cylinder at first looks like symmetrical situation. And indeed in a low Reynolds number it is a steady state. However after a certain value of Reynolds number, vortexes are created in an infinite parade and this phenomenon is called Von Karman vortex street (see Figure 9.4) which named after Von Karman. These vortexes are created in a non–symmetrical way and hence create an unsteady situation. When Reynolds number increases, these vortexes are mixed and the flow becomes turbulent which, can be considered a steady state[16].

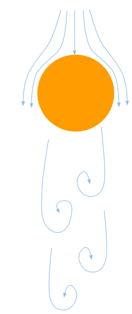

Fig. -9.4. Oscillating Von Karman Vortex Street.

The pressure P is the pressure at infinity or when the velocity is at rest. c is the speed of sound of the fluid at rest or characteristic value. The value of the viscosity, μ is typically some kind averaged value. The inability to define a fix value leads also to new dimensionless numbers which represent the deviations of these properties.

9.4.1 The Significance of these Dimensionless Numbers

Reynolds number, named in the honor of Reynolds, represents the ratio of the momentum forces to the viscous forces. Historically, this number was one of the first numbers to be introduced to fluid mechanics. This number determines, in many cases, the flow regime.

Example 9.13:

Eckert number[17] determines whether the role of the momentum energy is transferred to thermal energy is significant to affect the flow. This effect is important in situations where high speed is involved. This fact suggests that Eckert number is related to Mach number. Determine this relationship and under what circumstances this relationship is true.

<u>Solution</u>

[16]This is an example where the more unsteady the situation becomes the situation can be analyzed as a steady state because averages have a significant importance.

[17]This example is based on Bird, Lightfoot and Stuart "Transport Phenomena".

In Table 9.8 Mach and Eckert numbers are defined as

$$Ec = \frac{U^2}{C_p\,\Delta T} \qquad\qquad M = \frac{U}{\sqrt{\dfrac{P}{\rho}}} \tag{9.XIII.a}$$

The material which obeys the ideal flow model[18] ($P/\rho = R\,T$ and $P = C_1\,\rho^k$) can be written that

$$M = U \Bigg/ \sqrt{\frac{P}{\rho}} = \frac{U}{\sqrt{k\,R\,T}} \tag{9.XIII.b}$$

For the comparison, the reference temperature used to be equal to zero. Thus Eckert number can be written as

$$\sqrt{Ec} = \frac{U}{\sqrt{C_p\,T}} = \frac{U}{\sqrt{\underbrace{\left(\dfrac{R\,k}{k-1}\right)}_{C_p} T}} = \frac{\sqrt{k-1}\,U}{\sqrt{k\,R\,T}} = \sqrt{k-1}\,M \tag{9.XIII.c}$$

The Eckert number and Mach number are related under ideal gas model and isentropic relationship.

End Solution

Brinkman number measures of the importance of the viscous heating relative the conductive heat transfer. This number is important in cases when a large velocity change occurs over short distances such as lubricant, supersonic flow in rocket mechanics creating large heat effect in the head due to large velocity (in many place it is a combination of Eckert number with Brinkman number. The Mach number is based on different equations depending on the property of the medium in which pressure disturbance moves through. Cauchy number and Mach number are related as well and see Example 9.15 for explanation.

Example 9.14:

For historical reason some fields prefer to use certain numbers and not other ones. For example in Mechanical engineers prefer to use the combination Re and We number while Chemical engineers prefers to use the combination of Re and the Capillary number. While in some instances this combination is justified, other cases it is arbitrary. Show what the relationship between these dimensionless numbers.

Solution

The definitions of these number in Table 9.8

$$We = \frac{\rho\,U^2\,\ell}{\sigma} \qquad Re = \frac{\rho\,U\,\ell}{\mu} \qquad Ca = \frac{\mu\,U}{\sigma} = \frac{U}{\dfrac{\sigma}{\mu}} \tag{9.XIV.a}$$

[18] See for more details http://www.potto.org/gasDynamics/node70.html

Dividing Weber number by Reynolds number yields

$$\frac{We}{Re} = \frac{\dfrac{\rho\, U^2\, \ell}{\sigma}}{\dfrac{\rho\, U\, \ell}{\mu}} = \frac{U}{\dfrac{\sigma}{\mu}} = Ca \tag{9.XIV.b}$$

End Solution

Euler number is named after Leonhard Euler (1707 1783), a German Physicist who pioneered so many fields that it is hard to say what and where are his greatest contributions. Eulers number and Cavitation number are essentially the same with the exception that these numbers represent different driving pressure differences. This difference from dimensional analysis is minimal. Furthermore, Euler number is referred to as the pressure coefficient, C_p. This confusion arises in dimensional analysis because historical reasons and the main focus area. The cavitation number is used in the study of cavitation phenomena while Euler number is mainly used in calculation of resistances.

Example 9.15:

Explained under what conditions and what are relationship between the Mach number and Cauchy number?

<u>Solution</u>

Cauchy number is defined as

$$Cau = \frac{\rho \boldsymbol{U}^2}{E} \tag{9.XV.a}$$

The square root of Cauchy number is

$$\sqrt{Cau} = \frac{U}{\sqrt{\dfrac{E}{\rho}}} \tag{9.XV.b}$$

In the liquid phase the speed of sound is approximated as

$$c = \frac{E}{\rho} \tag{9.XV.c}$$

Using equation (9.XV.b) transforms equation (9.XV.a) into

$$\sqrt{Cau} = \frac{U}{c} = M \tag{9.49}$$

Thus the square root of Cau is equal to Mach number in the liquid phase. In the solid phase equation (9.XV.c) is less accurate and speed of sound depends on the direction of the grains. However, as first approximation, this analysis can be applied also to the solid phase.

End Solution

9.4.2 Relationship Between Dimensionless Numbers

The Dimensionless numbers since many of them have formulated in a certain field tend to be duplicated. For example, the Bond number is referred in Europe as Eotvos number. In addition to the above confusion, many dimensional numbers expressed the same things under certain conditions. For example, Mach number and Eckert Number under certain circumstances are same.

Example 9.16:
Galileo Number is a dimensionless number which represents the ratio of gravitational forces and viscous forces in the system as

$$Ga = \frac{\rho^2 \, g \, \ell^3}{\mu^2} \tag{9.XVI.a}$$

The definition of Reynolds number has viscous forces and the definition of Froude number has gravitational forces. What are the relation between these numbers?

Example 9.17:
Laplace Number is another dimensionless number that appears in fluid mechanics which related to Capillary number. The Laplace number definition is

$$La = \frac{\rho \, \sigma \, \ell}{\mu^2} \tag{9.XVII.a}$$

Show what are the relationships between Reynolds number, Weber number and Laplace number.

Example 9.18:
The Rotating Froude Number is a somewhat a similar number to the regular Froude number. This number is defined as

$$Fr_R = \frac{\omega^2 \, \ell}{g} \tag{9.XVIII.a}$$

What is the relationship between two Froude numbers?

Example 9.19:
Ohnesorge Number is another dimensionless parameter that deals with surface tension and is similar to Capillary number and it is defined as

$$Oh = \frac{\mu}{\sqrt{\rho \, \sigma \, \ell}} \tag{9.XIX.a}$$

Defined Oh in term of We and Re numbers.

9.4.3 Examples for Dimensional Analysis

Example 9.20:

The similarity of pumps is determined by comparing several dimensional numbers among them are Reynolds number, Euler number, Rossby number etc. Assume that the only numbers which affect the flow are Reynolds and Euler number. The flow rate of the imaginary pump is 0.25 [m^3/sec] and pressure increase for this flow rate is 2 [Bar] with 2500 [kw]. Due to increase of demand, it is suggested to replace the pump with a 4 times larger pump. What is the new estimated flow rate, pressure increase, and power consumption?

Solution

It provided that the Reynolds number controls the situation. The density and viscosity remains the same and hence

$$Re_m = Re_p \Longrightarrow U_m\, D_m = U_p\, D_p \Longrightarrow U_p = \frac{D_m}{D_P} U_m \tag{9.XX.a}$$

It can be noticed that initial situation is considered as the model and while the new pump is the prototype. The new flow rate, Q, depends on the ratio of the area and velocity as

$$\frac{Q_p}{Q_m} = \frac{A_p\, U_p}{A_m\, U_m} \Longrightarrow Q_p = Q_m \frac{A_p\, U_p}{A_m\, U_m} = Q_m \frac{{D_p}^2\, U_p}{{D_m}^2\, U_m} \tag{9.XX.b}$$

Thus the prototype flow rate is

$$Q_p = Q_m \left(\frac{D_p}{D_m}\right)^3 = 0.25 \times 4^3 = 16 \left[\frac{m^3}{sec}\right] \tag{9.XX.c}$$

The new pressure is obtain by comparing the Euler number as

$$Eu_p = Eu_m \Longrightarrow \left(\frac{\Delta P}{\frac{1}{2}\rho\, U^2}\right)_p = \left(\frac{\Delta P}{\frac{1}{2}\rho\, U^2}\right)_m \tag{9.XX.d}$$

Rearranging equation (9.XX.d) provides

$$\frac{(\Delta P)_p}{(\Delta P)_m} = \frac{(\not\rho\, U^2)_p}{(\not\rho\, U^2)_m} = \frac{(U^2)_p}{(U^2)_m} \tag{9.XX.e}$$

Utilizing equation (9.XX.a)

$$\Delta P_p = \Delta P_m \left(\frac{D_p}{D_m}\right)^2 \tag{9.XX.f}$$

The power can be obtained from the following

$$\dot{W} = \frac{F\,\ell}{t} = F\,U = P\,A\,U \tag{9.XX.g}$$

In this analysis, it is assumed that pressure is uniform in the cross section. This assumption is appropriate because only the secondary flows in the radial direction (to be discussed in this book section on pumps.). Hence, the ratio of power between the two pump can be written as

$$\frac{\dot{W}_p}{\dot{W}_m} = \frac{(P\,A\,U)_p}{(P\,A\,U)_m} \tag{9.XX.h}$$

Utilizing equations above in this ratio leads to

$$\frac{\dot{W}_p}{\dot{W}_m} = \overbrace{\left(\frac{D_p}{D_m}\right)^2}^{P_p/P_m} \overbrace{\left(\frac{D_p}{D_m}\right)^2}^{A_p/A_m} \overbrace{\left(\frac{D_p}{D_m}\right)}^{U_p/U_m} = \left(\frac{D_p}{D_m}\right)^5 \tag{9.XX.i}$$

End Solution

Example 9.21:

The flow resistance to flow of the water in a pipe is to be simulated by flow of air. Estimate the pressure loss ratio if Reynolds number remains constant. This kind of study appears in the industry in which the compressibility of the air is ignored. However, the air is a compressible substance that flows the ideal gas model. Water is a substance that can be considered incompressible flow for relatively small pressure change. Estimate the error using the averaged properties of the air.

SOLUTION

For the first part, the Reynolds number is the single controlling parameter which affects the pressure loss. Thus it can be written that the Euler number is function of the Reynolds number.

$$Eu = f(Re) \tag{9.XXI.a}$$

Thus, to have a similar situation the Reynolds and Euler have to be same.

$$Re_p = Re_m \qquad\qquad Eu_m = Eu_p \tag{9.XXI.b}$$

Hence,

$$\frac{U_m}{U_p} = \frac{\ell_p}{\ell_m}\frac{\rho}{\rho_m}\frac{\mu_p}{\mu_m} \tag{9.XXI.c}$$

and for Euler number

$$\frac{\Delta P_m}{\Delta P_p} = \frac{\rho_m}{\rho_p}\frac{U_m}{U_p} \tag{9.XXI.d}$$

and utilizing equation (9.XXI.c) yields

$$\frac{\Delta P_m}{\Delta P_p} = \left(\frac{\ell_p}{\ell_m}\right)^2 \left(\frac{\mu_m}{\mu_p}\right)^2 \left(\frac{\rho_p}{\rho_m}\right) \tag{9.XXI.e}$$

Inserting the numerical values results in

$$\frac{\Delta P_m}{\Delta P_p} = 1 \times 1000 \times \tag{9.XXI.f}$$

It can be noticed that the density of the air changes considerably hence the calculations produce a considerable error which can render the calculations useless (a typical problem of Buckingham's method). Assuming a new variable that effect the problem, air density variation. If that variable is introduced into problem, air can be used to simulate water flow. However as a first approximation, the air properties are calculated based on the averaged values between the entrance and exit values. If the pressure reduction is a function of pressure reduction (iterative process).

to be continue

End Solution

Example 9.22:

A device operating on a surface of a liquid to study using a model with a ratio 1:20. What should be ratio of kinematic viscosity between the model and prototype so that Froude and Reynolds numbers remain the same. Assume that body force remains the same and velocity is reduced by half.

<u>SOLUTION</u>

The requirement is that Reynolds

$$Re_m = R_p \Longrightarrow \left(\frac{U\,\ell}{\nu}\right)_p = \left(\frac{U\,\ell}{\nu}\right)_m \tag{9.XXII.a}$$

The Froude needs to be similar so

$$Fr_m = Fr_p \Longrightarrow \left(\frac{U}{\sqrt{g\,\ell}}\right)_p = \left(\frac{U\,\ell}{\nu}\right)_m \tag{9.XXII.b}$$

dividing equation (9.XXII.a) by equation (9.XXII.b) results in

$$\left(\frac{U\,\ell}{\nu}\right)_p \Big/ \left(\frac{U}{\sqrt{g\,\ell}}\right)_p = \left(\frac{U\,\ell}{\nu}\right)_m \Big/ \left(\frac{U}{\sqrt{g\,\ell}}\right)_m \tag{9.XXII.c}$$

or

$$\left(\frac{\ell\sqrt{g\,\ell}}{\nu}\right)_p = \left(\frac{\ell\sqrt{g\,\ell}}{\nu}\right)_m \tag{9.XXII.d}$$

If the body force[19], g, The kinematic viscosity ratio is then

$$\frac{\nu_p}{\nu_m} = \left(\frac{\ell_m}{\ell_p}\right)^{3/2} = (1/20)^{3/2} \tag{9.XXII.e}$$

[19] The body force does not necessarily have to be the gravity.

It can be noticed that this can be achieved using Ohnesorge Number like this presentation.

End Solution

9.5 Summary

The two dimensional analysis methods or approaches were presented in this chapter. Buckingham's π technique is a quick "fix approach" which allow rough estimates and relationship between model and prototype. Nusselt's approach provides an heavy duties approach to examine what dimensionless parameters effect the problem. It can be shown that these two techniques in some situations provide almost similar solution. In other cases, these technique proves different and even conflicting results. The dimensional analysis technique provides a way to simplify models (solving the governing equation by experimental means) and to predict effecting parameters.

9.6 Appendix summary of Dimensionless Form of Navier–Stokes Equations

In a vector form Navier–Stokes equations can be written and later can be transformed into dimensionless form which will yield dimensionless parameters. First, the typical or characteristics values of scaling parameters has to be presented and appear in the following table

Parameter Symbol	Parameter Description	Units
h	characteristic length	$[L]$
U_0	characteristic velocity	$\left[\dfrac{L}{t}\right]$
f	characteristic frequency	$\left[\dfrac{1}{t}\right]$
ρ_0	characteristic density	$\left[\dfrac{M}{L^3}\right]$
$P_{max} - P_{\infty}$	maximum pressure drive	$\left[\dfrac{M}{L\,t^2}\right]$

Basic non–dimensional form of the parameters

$$\begin{array}{ccc} \tilde{t} = ft & \tilde{\boldsymbol{r}} = \dfrac{\vec{\boldsymbol{r}}}{h} & \tilde{\boldsymbol{U}} = \dfrac{\vec{\boldsymbol{U}}}{U_0} \\ \tilde{\boldsymbol{P}} = \dfrac{\boldsymbol{P} - P_{\infty}}{P_{max} - P_{\infty}} & \tilde{\boldsymbol{\nabla}} = h\,\boldsymbol{\nabla} & \tilde{\boldsymbol{\rho}} = \dfrac{\rho}{\rho_0} \end{array} \tag{9.50}$$

For the Continuity Equation (8.17) for non–compressible substance can be transformed into

$$\cancelto{0}{\frac{\partial \rho}{\partial t}} + \nabla \cdot (\tilde{\boldsymbol{\rho}} \boldsymbol{U}) = 0 \tag{9.51}$$

For the N-S equation, every additive term has primary dimensions $m^1 L^{-2} t^{-2}$. To non nondimensionalization, we multiply every term by $L/(V^2)$, which has primary dimensions $m^{-1} L^2 t^2$, so that the dimensions cancel.

Using these definitions equation (8.111) results in

$$\frac{f\,h}{U_0} \frac{\partial \tilde{\boldsymbol{U}}}{\partial \tilde{t}} + \left(\tilde{\boldsymbol{U}} \cdot \tilde{\boldsymbol{\nabla}}\right) \tilde{\boldsymbol{U}} = -\left(\frac{P_{max} - P_\infty}{\rho \tilde{\boldsymbol{U}}}\right) \tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{P}} + \frac{1}{\dfrac{\tilde{\boldsymbol{U}}^2}{g\,h}} \vec{f}_g + \frac{1}{\dfrac{\rho\, \tilde{\boldsymbol{U}}\, h}{\mu}} \tilde{\boldsymbol{\nabla}}^2 \tilde{\boldsymbol{U}} \tag{9.52}$$

Or after using the definition of the dimensionless parameters as

$$St\, \frac{\partial \tilde{\boldsymbol{U}}}{\partial \tilde{t}} + \left(\tilde{\boldsymbol{U}} \cdot \tilde{\boldsymbol{\nabla}}\right) \tilde{\boldsymbol{U}} = -Eu \tilde{\boldsymbol{\nabla}} \tilde{\boldsymbol{P}} + \frac{1}{Fr^2} \vec{f}_g + \frac{1}{Re} \tilde{\boldsymbol{\nabla}}^2 \tilde{\boldsymbol{U}} \tag{9.53}$$

The definition of Froude number is not consistent in the literature. In some places Fr is defined as the square of $Fr = U^2/g\,h$.

The Strouhal number is named after Vincenz Strouhal (1850 1922), who used this parameter in his study of "singing wires." This parameter is important in unsteady, oscillating flow problems in which the frequency of the oscillation is important.

Example 9.23:

A device is accelerated linearly by a constant value $\boldsymbol{B}$*. Write a new N–S and continuity equations for incompressible substance in the a coordinate system attached to the body. Using these equations developed new dimensionless equations so the new "Froude number" will contain or "swallow" by the new acceleration. Measurement has shown that the acceleration to be constant with small sinusoidal on top the constant such away as*

$$\boldsymbol{a} = \boldsymbol{B} + \boldsymbol{\epsilon} \sin\left(\frac{f}{2\,\pi}\right) \tag{9.XXIII.a}$$

Suggest a dimensionless parameter that will take this change into account.

Supplemental Problems

1. An airplane wing of chord length 3 [m] moves through still air at 15°Cand 1 $[Bar]$ and at at a speed of 15 $[m/sec]$. What is the air velocity for a 1:20 scale model to achieve dynamic similarity between model and prototype? Assume that in the model the air has the same pressure and temperature as that in prototype. If the

air is considered as compressible, what velocity is required for pressure is 1.5$[bar]$ and temperature 20°C? What is the required velocity of the air in the model test when the medium is made of water to keep the dynamic similarity?

2. An airplane 100$[m]$ long is tested by 1 $[m]$ model. If the airplane velocity is 120 $[m]$ and velocity at the wind–tunnel is 60 $[m]$, calculate the model and the airplane Reynolds numbers. You can assume that both model and prototype working conditions are the same (1$[Bar]$ and 60°C).

3. What is the pipe diameter for oil flowing at speed of 1$[m/sec]$ to obtain dynamic similarity with a pipe for water flowing at 3 $[m/sec]$ in a 0.02$[m]$ pipe. State your assumptions.

4. The pressure drop for water flowing at 1 $[m/sec]$ in a pipe was measured to be 1 $[Bar]$. The pipe is 0.05 $[m]$ diameter and 100 $[m]$ in length. What should be velocity of Castor oil to get the same Reynolds number? What would be pressure drop in that case?

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A micro-biography of Edgar Buckingham

Edgar Buckingham (18671940) was educated at Harvard and Leipzig, and worked at the (US) National Bureau of Standards (now the National Institute of Standards and Technology, or NIST) 1905–1937. His fields of expertise included soil physics, gas properties, acoustics, fluid mechanics, and black–body radiation.