Note:

CHAPTER 8: ENERGY

Version: 0.3.1.1 December 21, 2011

This chapter is part of the textbook:

"Basics of Fluid Mechanics"

You can download the whole book if you like from: *www.potto.org.*

Please be aware that this book is updated frequently — every three weeks or so.

GENICK BAR-MEIR, PH.D.
CHICAGO, ILLINOIS
DECEMBER 21, 2011

THE LIST OF THE AVAILABLE BOOKS IN POTTO PROJECT

Project Name	Progress	Remarks	Version	Availability for Public Download
Compressible Flow	beta		0.4.8.6	✔
Gas Dynamics Tables	final	World biggest	1.2	✔
Die Casting	alpha		0.1.2	✔
Dynamics	NSY		0.0.0	✖
Fluid Mechanics	beta		0.2.9	✔
Heat Transfer	NSY	Based on Eckert	0.0.0	✖
Mechanics	NSY		0.0.0	✖
Open Channel Flow	NSY		0.0.0	✖
Statics	early alpha	first chapter	0.0.1	✖
Strength of Material	NSY		0.0.0	✖
Thermodynamics	early alpha		0.0.01	✖
Two/Multi phases flow	NSY	Tel-Aviv's notes	0.0.0	✖

NSY = Not Started Yet

CHAPTER 7

Energy Conservation

7.1 The First Law of Thermodynamics

This chapter focuses on the energy conservation which is the first law of thermodynamics[1]. The fluid, as all phases and materials, obeys this law which creates strange and wonderful phenomena such as a shock and choked flow. Moreover, this law allows to solve problems, which were assumed in the previous chapters. For example, the relationship between height and flow rate was assumed previously, here it will be derived. Additionally a discussion on various energy approximation is presented.

It was shown in Chapter 2 that the energy rate equation (2.10) for a system is

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + \frac{D\left(m\,U^2\right)}{Dt} + \frac{D\left(m\,g\,z\right)}{Dt} \tag{7.1}$$

This equation can be rearranged to be

$$\dot{Q} - \dot{W} = \frac{D}{Dt}\left(E_U + m\,\frac{U^2}{2} + m\,g\,z\right) \tag{7.2}$$

Equation (7.2) is similar to equation (6.3) in which the right hand side has to be interpreted and the left hand side interpolated using the Reynold's Transport Theorem (RTT)[2]. The right hand side is very complicated and only some of the effects will be discussed (It is only an introductory material).

[1]Thermodynamics is the favorite topic of this author since it was his major in high school. Clearly this topic is very important and will be extensively discussed here. However, during time of the constructing this book only a simple skeleton by Potto standards will be build.

[2]Some view the right hand side as external effects while the left side of the equation represents the internal effects. This simplistic representation is correct only under extreme conditions. For example, the above view is wrong when the heat convection, which is external force, is included on the right hand side.

The energy transfer is carried (mostly[3]) by heat transfer to the system or the control volume. There are three modes of heat transfer, conduction, convection[4] and radiation. In most problems, the radiation is minimal. Hence, the discussion here will be restricted to convection and conduction. Issues related to radiation are very complicated and considered advance material and hence will be left out. The issues of convection are mostly covered by the terms on the left hand side. The main heat transfer mode on the left hand side is conduction. Conduction for most simple cases is governed by Fourier's Law which is

$$d\dot{q} = k_T \frac{dT}{dn} dA \tag{7.3}$$

Where $d\dot{q}$ is heat transfer to an infinitesimal small area per time and k_T is the heat conduction coefficient. The heat derivative is normalized into area direction. The total heat transfer to the control volume is

$$\dot{Q} = \int_{A_{cv}} k \frac{dT}{dn} dA \tag{7.4}$$

The work done on the system is more complicated to express than the heat transfer. There are two kinds of works that the system does on the surroundings. The first kind work is by the friction or the shear stress and the second by normal force. As in the previous chapter, the surface forces are divided into two categories: one perpendicular to the surface and one with the surface direction. The work done by system on the surroundings (see Figure 7.1) is

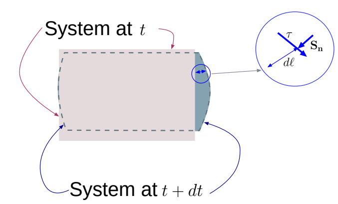


Fig. -7.1. The work on the control volume is done by two different mechanisms, $\mathbf{S_n}$ *and* τ.

$$dw = \overbrace{-\boldsymbol{S}\, d\boldsymbol{A}}^{d\boldsymbol{F}} \cdot d\ell = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \overbrace{d\boldsymbol{\ell} dA}^{dV} \tag{7.5}$$

The change of the work for an infinitesimal time (excluding the shaft work) is

$$\frac{dw}{dt} = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \overbrace{\frac{d\boldsymbol{\ell}}{dt}}^{U} dA = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \boldsymbol{U}\, dA \tag{7.6}$$

The total work for the system including the shaft work is

$$\dot{W} = -\int_{Ac.v.} \left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \boldsymbol{U}\, dA - W_{shaft} \tag{7.7}$$

[3]There are other methods such as magnetic fields (like microwave) which are not part of this book.

[4]When dealing with convection, actual mass transfer must occur and thus no convection is possible to a system by the definition of system.

The energy equation (7.2) for system is

$$\begin{aligned}\int_{A_{sys}} k_T \frac{dT}{dn} dA + & \int_{A_{sys}} \left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) dV \\ & + \dot{W}_{shaft} = \frac{D}{Dt}\int_{V_{sys}} \rho \left(E_U + m\frac{U^2}{2} + g\,z\right) dV\end{aligned} \tag{7.8}$$

Equation (7.8) does not apply any restrictions on the system. The system can contain solid parts as well several different kinds of fluids. Now Reynolds Transport Theorem can be used to transformed the left hand side of equation (7.8) and thus yields

Energy Equation

$$\begin{aligned}\int_{A_{cv}} k_T \frac{dT}{dn} dA + & \int_{A_{cv}} \left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) dA + \dot{W}_{shaft} = \\ & \frac{d}{dt}\int_{V_{cv}} \rho \left(E_u + m\frac{U^2}{2} + g\,z\right) dV \\ & + \int_{A_{cv}} \left(E_u + m\frac{U^2}{2} + g\,z\right) \rho\, U_{rn} dA\end{aligned} \tag{7.9}$$

From now on the notation of the control volume and system will be dropped since all equations deals with the control volume. In the last term in equation (7.9) the velocity appears twice. Note that U is the velocity in the frame of reference while U_{rn} is the velocity relative to the boundary. As it was discussed in the previous chapter the normal stress component is replaced by the pressure (see equation (6.8) for more details). The work rate (excluding the shaft work) is

$$\dot{W} \cong \overbrace{\int_S P\hat{n}\cdot\boldsymbol{U} dA}^{\text{flow work}} - \int_S \boldsymbol{\tau}\cdot\boldsymbol{U}\,\hat{n}\,dA \tag{7.10}$$

The first term on the right hand side is referred to in the literature as the flow work and is

$$\int_S P\hat{n}\cdot\boldsymbol{U} dA = \int_S P\,\overbrace{\left(U - U_b\right)\hat{n}}^{U_{rn}}\,dA + \int_S P\,U_{bn} dA \tag{7.11}$$

Equation (7.11) can be further manipulated to become

$$\int_S P\hat{n}\cdot\boldsymbol{U} dA = \overbrace{\int_S \frac{P}{\rho}\,\rho\,U_{rn}\,dA}^{\text{work due to the flow}} + \overbrace{\int_S P U_{bn} dA}^{\text{work due to boundaries movement}} \tag{7.12}$$

The second term is referred to as the shear work and is defined as

$$\dot{W}_{shear} = -\int_S \boldsymbol{\tau} \cdot \boldsymbol{U} dA \tag{7.13}$$

Substituting all these terms into the governing equation yields

$$\begin{aligned}\dot{Q} - \dot{W}_{shear} - \quad \dot{W}_{shaft} = \frac{d}{dt}\int_V \left(E_u + \frac{U^2}{2} + g\,z\right) dV + \\ \int_S \left(E_u + \frac{P}{\rho} + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{rn} dA\end{aligned} \tag{7.14}$$

The new term P/ρ combined with the internal energy, E_u is referred to as the enthalpy, h, which was discussed on page 50. With these definitions equation (7.14) transformed

Simplified Energy Equation

$$\begin{aligned}\dot{Q} - \dot{W}_{shear} + \quad \dot{W}_{shaft} = \frac{d}{dt}\int_V \left(E_u + \frac{U^2}{2} + g\,z\right) \rho\,dV + \\ \int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{bn} dA\end{aligned} \tag{7.15}$$

Equation (7.15) describes the energy conservation for the control volume in stationary coordinates. Also note that the shear work inside the the control volume considered as shaft work.

The example of flow from a tank or container is presented to demonstrate how to treat some of terms in equation (7.15).

Flow Out From A Container

In the previous chapters of this book, the flow rate out of a tank or container was assumed to be a linear function of the height. The flow out is related to the height but in a more complicate function and is the focus of this discussion. The energy equation with mass conservation will be utilized for this analysis. In this analysis several assumptions are made which includes the following: constant density, the gas density is very small compared to liquid density, and exit area is relatively small, so the velocity can be assumed uniform (not a function of the opening height)[5], surface tension effects are negligible and

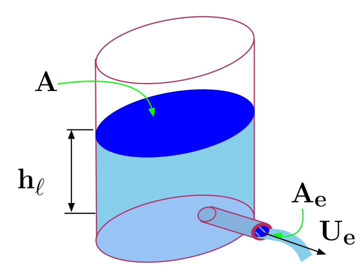


Fig. -7.2. Discharge from a Large Container with a small diameter.

[5]Later a discussion about the height opening effects will be discussed.

the liquid surface is straight[6]. Additionally, the temperature is assumed to constant. The control volume is chosen so that all the liquid is included up to exit of the pipe. The conservation of the mass is

$$\frac{d}{dt}\int_V \not\rho \, dV + \int_A \not\rho \, U_{rn} \, dA = 0 \tag{7.16}$$

which also can be written (because $\frac{d\rho}{dt} = 0$) as

$$\int_A U_{bn} \, dA + \int_A U_{rn} dA = 0 \tag{7.17}$$

Equation (7.17) provides the relationship between boundary velocity to the exit velocity as

$$A \, U_b = A_e \, U_e \tag{7.18}$$

Note that the boundary velocity is not the averaged velocity but the actual velocity. The averaged velocity in z direction is same as the boundary velocity

$$U_b = U_z = \frac{dh}{dt} = \frac{A_e}{A} \, U_e \tag{7.19}$$

The x component of the averaged velocity is a function of the geometry and was calculated in Example 5.12 to be larger than

$$\overline{U_x} \precapprox \frac{2\,r}{h}\frac{A_e}{A} U_e \Longrightarrow \overline{U_x} \cong \frac{2\,r}{h} \, U_b = \frac{2\,r}{h}\,\frac{dh}{dt} \tag{7.20}$$

In this analysis, for simplicity, this quantity will be used.

The averaged velocity in the y direction is zero because the flow is symmetrical[7]. However, the change of the kinetic energy due to the change in the velocity field isn't zero. The kinetic energy of the tank or container is based on the half part as shown in Figure 7.3. Similar estimate that was done for x direction can be done to every side of the opening if they are not symmetrical. Since in this case the geometry is assumed to be symmetrical one side is sufficient as

$$\overline{U_y} \cong \frac{(\pi - 2) r}{8\,h}\,\frac{dh}{dt} \tag{7.21}$$

[6]This assumption is appropriated only under certain conditions which include the geometry of the tank or container and the liquid properties. A discussion about this issue will be presented in the Dimensional Chapter and is out of the scope of this chapter. Also note that the straight surface assumption is not the same surface tension effects zero.

Also notice that the surface velocity is not zero. The surface has three velocity components which non have them vanish. However, in this discussion it is assumed that surface has only one component in z direction. Hence it requires that velocity profile in x y to be parabolic. Second reason for this exercise the surface velocity has only one component is to avoid dealing with Bar-Meir's instability.

[7]For the mass conservation analysis, the velocity is zero for symmetrical geometry and some other geometries. However, for the energy analysis the averaged velocity cannot be considered zero.

The energy balance can be expressed by equation (7.15) which is applicable to this case. The temperature is constant[8]. In this light, the following approximation can be written

$$\dot{Q} = \frac{E_u}{dt} = h_{in} - h_{out} = 0 \tag{7.22}$$

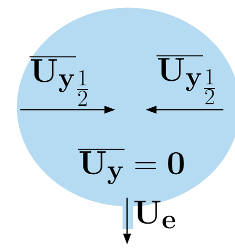


Fig. -7.3. How to compensate and estimate the kinetic energy when averaged Velocity is zero.

The boundary shear work is zero because the velocity at tank boundary or walls is zero. Furthermore, the shear stresses at the exit are normal to the flow direction hence the shear work is vanished. At the free surface the velocity has only normal component[9] and thus shear work vanishes there as well. Additionally, the internal shear work is assumed negligible.

$$\dot{W}_{shear} = \dot{W}_{shaft} = 0 \tag{7.23}$$

Now the energy equation deals with no "external" effects. Note that the (exit) velocity on the upper surface is zero $U_{rn} = 0$.

Combining all these information results in

$$\overbrace{\frac{d}{dt}\int_V \left(\frac{U^2}{2} + g\,z\right)\rho\,dV}^{\text{internal energy change}} + \overbrace{\overbrace{\int_A \left(\frac{P_e}{\rho} + \frac{{U_e}^2}{2}\right) U_e\,\rho\,dA}^{\text{energy in and out}} - \overbrace{\int_A P_a\,U_b\,dA}^{\text{upper surface work}}}^{\text{energy flow out}} = 0 \tag{7.24}$$

Where U_b is the upper boundary velocity, P_a is the external pressure and P_e is the exit pressure[10].

The pressure terms in equation (7.24) are

$$\int_A \frac{P_e}{\rho}\,U_e\,\rho dA - \int_A P_a\,U_b\,dA = P_e\int_A U_e\,dA - P_a\int_A U_b\,dA \tag{7.25}$$

It can be noticed that $P_a = P_e$ hence

$$P_a \overbrace{\left(\int_A U_e\,dA - \int_A U_b\,dA\right)}^{=0} = 0 \tag{7.26}$$

[8]This approach is a common approximation. Yet, why this approach is correct in most cases is not explained here. Clearly, the dissipation creates a loss that has temperature component. In this case, this change is a function of Eckert number, Ec which is very small. The dissipation can be neglected for small Ec number. Ec number is named after this author's adviser, E.R.G. Eckert. A discussion about this effect will be presented in the dimensional analysis chapter. Some examples how to calculate these losses will be resent later on.

[9]It is only the same assumption discussed earlier.

[10]It is assumed that the pressure in exit across section is uniform and equal surroundings pressure.

The governing equation (7.24) is reduced to

$$\frac{d}{dt}\int_V \left(\frac{U^2}{2} + g\,z\right)\rho\,dV - \int_A \left(\frac{{U_e}^2}{2}\right) U_e\,\rho\,dA = 0 \tag{7.27}$$

The minus sign is because the flow is out of the control volume.

Similarly to the previous chapter which the integral momentum will be replaced by some kind of average. The terms under the time derivative can be divided into two terms as

$$\frac{d}{dt}\int_V \left(\frac{U^2}{2} + g\,z\right)\rho dV = \frac{d}{dt}\int_V \frac{U^2}{2}\,dV + \frac{d}{dt}\int_V g\,z\,\rho\,dV \tag{7.28}$$

The second integral (in the r.h.s) of equation (7.28) is

$$\frac{d}{dt}\int_V g\,z\,\rho\,dV = g\,\rho\,\frac{d}{dt}\int_A\int_0^h z\,\overbrace{dz\,dA}^{dV} \tag{7.29}$$

Where h is the height or the distance from the surface to exit. The inside integral can be evaluated as

$$\int_0^h zdz = \frac{h^2}{2} \tag{7.30}$$

Substituting the results of equation (7.30) into equation (7.29) yields

$$g\,\rho\,\frac{d}{dt}\int_A \frac{h^2}{2}\,dA = g\,\rho\,\frac{d}{dt}\left(\frac{h}{2}\,\overbrace{h\,A}^{V}\right) = g\,\rho\,A\,h\,\frac{d\,h}{dt} \tag{7.31}$$

The kinetic energy related to the averaged velocity with a correction factor which depends on the geometry and the velocity profile. Furthermore, Even the averaged velocity is zero the kinetic energy is not zero and another method should be used.

A discussion on the correction factor is presented to provide a better "averaged" velocity. A comparison between the actual kinetic energy and the kinetic energy due to the "averaged" velocity (to be called the averaged kinetic energy) provides a correction coefficient. The first integral can be estimated by examining the velocity profile effects. The averaged velocity is

$$U_{ave} = \frac{1}{V}\int_V U dV \tag{7.32}$$

The total kinetic energy for the averaged velocity is

$$\rho\,{U_{ave}}^2\,V = \rho\left(\frac{1}{V}\int_V U dV\right)^2 V = \rho\left(\int_V U dV\right)^2 \tag{7.33}$$

The general correction factor is the ratio of the above value to the actual kinetic energy as

$$C_F = \frac{\left(\displaystyle\int_V \rho\, U\, dV\right)^2}{\displaystyle\int_V \rho\, U^2\, dV} \neq \frac{\not\rho\, \left(U_{ave}\right)^2\, V}{\displaystyle\int_V \not\rho\, U^2\, dV} \tag{7.34}$$

Here, C_F is the correction coefficient. Note, the inequality sign because the density distribution for compressible fluid. The correction factor for a constant density fluid is

$$C_F = \frac{\left(\displaystyle\int_V \rho\, U\, dV\right)^2}{\displaystyle\int_V \rho\, U^2\, dV} = \frac{\left(\not\rho \displaystyle\int_V U\, dV\right)^2}{\not\rho \displaystyle\int_V U^2\, dV} = \frac{{U_{ave}}^2\, V}{\displaystyle\int_V U^2\, dV} \tag{7.35}$$

This integral can be evaluated for any given velocity profile. A large family of velocity profiles is laminar or parabolic (for one directional flow)[11]. For a pipe geometry, the velocity is

$$U\left(\frac{r}{R}\right) = U\left(\bar{r}\right) = U_{max}\left(1 - \bar{r}^2\right) = 2\, U_{ave}\left(1 - \bar{r}^2\right) \tag{7.36}$$

It can be noticed that the velocity is presented as a function of the reduced radius[12]. The relationship between U_{max} to the averaged velocity, U_{ave} is obtained by using equation (7.32) which yields $1/2$.

Substituting equation (7.36) into equation (7.35) results

$$\frac{{U_{ave}}^2\, V}{\displaystyle\int_V U^2\, dV} = \frac{{U_{ave}}^2\, V}{\displaystyle\int_V \left(2\, U_{ave}\left(1 - \bar{r}^2\right)\right)^2\, dV} = \frac{{U_{ave}}^2\, V}{\dfrac{4\, {U_{ave}}^2\, \pi\, L\, R^2}{3}} = \frac{3}{4} \tag{7.37}$$

The correction factor for many other velocity profiles and other geometries can be smaller or larger than this value. For circular shape, a good guess number is about 1.1. In this case, for simplicity reason, it is assumed that the averaged velocity indeed represent the energy in the tank or container. Calculations according to this point can improve the accurately based on the above discussion.

The difference between the "averaged momentum" velocity and the "averaged kinetic" velocity is also due to the fact that energy is added for different directions while in the momentum case, different directions cancel each other out.

[11] Laminar flow is not necessarily implies that the flow velocity profile is parabolic. The flow is parabolic only when the flow is driven by pressure or gravity. More about this issue in the Differential Analysis Chapter.

[12] The advantage is described in the Dimensional Analysis Chapter.

The unsteady state term then obtains the form

$$\frac{d}{dt}\int_V \rho\left(\frac{U^2}{2}+g\,y\right)dV \quad \cong \rho\,\frac{d}{dt}\left(\left[\frac{\overline{U}^2}{2}+\frac{g\,h}{2}\right]\overbrace{h\,A}^{V}\right) \tag{7.38}$$

The relationship between the boundary velocity to the height (by definition) is

$$U_b = \frac{dh}{dt} \tag{7.39}$$

Therefore, the velocity in the z direction[13] is

$$U_z = \frac{dh}{dt} \tag{7.40}$$

$$U_e = \frac{A}{A_e}\frac{dh}{dt} = -U_b\frac{dh}{dt} \tag{7.41}$$

Combining all the three components of the velocity (Pythagorean Theorem) as

$$\overline{U}^2 \cong \overline{U_x}^2 + \overline{U_y}^2 + \overline{U_z}^2 \tag{7.42}$$

$$\overline{U}^2 \cong \left(\frac{(\pi-2)\,r}{8\,h}\frac{dh}{dt}\right)^2 + \left(\frac{(\pi-1)\,r}{4\,h}\frac{dh}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 \tag{7.43}$$

$$\overline{U} \cong \frac{dh}{dt}\,\overbrace{\sqrt{\left(\frac{(\pi-2)\,r}{8\,h}\right)^2 + \left(\frac{(\pi-1)\,r}{4\,h}\right)^2 + 1^2}}^{f(G)} \tag{7.44}$$

It can be noticed that $f(G)$ is a weak function of the height inverse. Analytical solution of the governing equation is possible including this effect of the height. However, the mathematical complication are enormous[14] and this effect is assumed negligible and the function to be constant.

[13] A similar point was provided in mass conservation Chapter 5. However, it easy can be proved by construction the same control volume. The reader is encouraged to do it to get acquainted with this concept.

[14] The solution, not the derivation, is about one page. It must be remembered that is effect extremely important in the later stages of the emptying of the tank. But in the same vain, some other effects have to be taken into account which were neglected in construction of this model such as upper surface shape.

The last term is

$$\int_A \frac{{U_e}^2}{2} U_e\, \rho\, dA = \frac{{U_e}^2}{2} U_e\, \rho\, A_e = \frac{1}{2}\left(\frac{dh}{dt}\frac{A}{A_e}\right)^2 U_e\, \rho\, A_e \tag{7.45}$$

Combining all the terms into equation (7.27) results in

$$\not\rho\, \frac{d}{dt}\left(\left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right] \overbrace{h\,A}^{V}\right) - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A}{A_e}\right)^2 U_e\, \not\rho\, A_e = 0 \tag{7.46}$$

taking the derivative of first term on l.h.s. results in

$$\frac{d}{dt}\left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right] h\,A + \left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right] A\,\frac{dh}{dt} - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A}{A_e}\right)^2 U_e\, A_e = 0 \tag{7.47}$$

Equation (7.47) can be rearranged and simplified and combined with mass conservation [15].

Advance material can be skipped

Dividing equation (7.46) by $U_e\, A_e$ and utilizing equation (7.40)

$$\frac{d}{dt}\left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right] \frac{h\,A}{U_e\,A_e} + \left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right] \overbrace{\not A\,\frac{\not dh}{dt}}^{A\,\frac{A_e}{A}\,U_e} - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A}{A_e}\right)^2 \not U_e\, \not A_e = 0 \tag{7.48}$$

Notice that $\overline{U} = U_b\, f(G)$ and thus

$$\overbrace{\overline{U}}^{f(G)\,U_b}\, \frac{d\overline{U}}{dt}\, \frac{h\,A}{U_e\,A_e} + \frac{g}{2}\frac{dh}{dt}\,\frac{h\,A}{U_e\,A_e} + \left[\frac{\overline{U}^2}{2} + \frac{g\,h}{2}\right] - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A}{A_e}\right)^2 = 0 \tag{7.49}$$

Further rearranging to eliminate the "flow rate" transforms to

$$f(G)\,h\,\frac{d\overline{U}}{dt}\left(\cancelto{1}{\frac{U_b\,A}{U_e\,A_e}}\right) + \frac{g\,h}{2}\,\cancelto{1}{\frac{\frac{dh}{dt}A}{U_e\,A_e}} + \left[\frac{f(G)^2}{2}\left(\frac{dh}{dt}\right)^2 + \frac{g\,h}{2}\right] - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A}{A_e}\right)^2 = 0 \tag{7.50}$$

$$f(G)^2\,h\,\frac{d^2h}{dt^2} + \frac{g\,h}{2} + \left[\frac{f(G)^2}{2}\left(\frac{dh}{dt}\right)^2 + \frac{g\,h}{2}\right] - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A}{A_e}\right)^2 = 0 \tag{7.51}$$

[15] This part can be skipped to end of "advanced material".

End Advance material

Combining the gh terms into one yields

$$f(G)^2\, h\, \frac{d^2h}{dt^2} + g\, h + \frac{1}{2} \left(\frac{dh}{dt}\right)^2 \left[f(G)^2 - \left(\frac{A}{A_e}\right)^2 \right] = 0 \tag{7.52}$$

Defining a new tank emptying parameter, T_e, as

$$T_e = \left(\frac{A}{f(G)\, A_e}\right)^2 \tag{7.53}$$

This parameter represents the characteristics of the tank which controls the emptying process. Dividing equation (7.52) by $f(G)^2$ and using this parameter, equation (7.52) after minor rearrangement transformed to

$$h \left(\frac{d^2h}{dt^2} + \frac{g\, {A_e}^2}{T_e\, A^2} \right) + \frac{1}{2} \left(\frac{dh}{dt}\right)^2 [1 - T_e] = 0 \tag{7.54}$$

The solution can either of these equations[16]

$$- \int \frac{dh}{\sqrt{\dfrac{(k_1\, T_e - 2\, k_1)\; e^{\ln(h)\, Te} + 2\, g\, h^2}{h\; (Te - 2)\; f(G)}}} = t + k_2 \tag{7.55}$$

or

$$\int \frac{dh}{\sqrt{\dfrac{(k_1\, T_e - 2\, k_1)\; e^{\ln(h)\, Te} + 2\, g\, h^2}{h\; (Te - 2)\; f(G)}}} = t + k_2 \tag{7.56}$$

The solution with the positive solution has no physical meaning because the height cannot increase with time. Thus define function of the height as

$$f(h) = - \int \frac{dh}{\sqrt{\dfrac{(k_1\, T_e - 2\, k_1)\; e^{\ln(h)\, Te} + 2\, g\, h^2}{h\; (Te - 2)\; f(G)}}} \tag{7.57}$$

The initial condition for this case are: one the height initial is

$$h(0) = h_0 \tag{7.58}$$

[16]A discussion about this equation appear in the mathematical appendix.

The initial boundary velocity is

$$\frac{dh}{dt} = 0 \tag{7.59}$$

This condition pose a physical limitation[17] which will be ignored. The first condition yields

$$k_2 = -f(h_0) \tag{7.60}$$

The second condition provides

$$\frac{dh}{dt} = 0 = \sqrt{\frac{\left(k_1\,T_e - 2\,k_1\right)\,e^{\ln(h_0)\,Te} + 2\,g\,{h_0}^2}{h_0\,\left(Te-2\right)\,f(G)}} \tag{7.61}$$

The complication of the above solution suggest a simplification in which

$$\frac{d^2h}{dt^2} << \frac{g\,{A_e}^2}{T_e\,A^2} \tag{7.62}$$

which reduces equation (7.54) into

$$h\left(\frac{g\,{A_e}^2}{T_e\,A^2}\right) + \frac{1}{2}\left(\frac{dh}{dt}\right)^2\left[1 - T_e\right] = 0 \tag{7.63}$$

While equation (7.63) is still non linear equation, the non linear element can be removed by taking negative branch (height reduction) of the equation as

$$\left(\frac{dh}{dt}\right)^2 = \frac{2\,g\,h}{-1 + \left(\frac{A}{A_e}\right)^2} \tag{7.64}$$

It can be noticed that T_e "disappeared" from the equation. And taking the "positive" branch

$$\frac{dh}{dt} = \frac{\sqrt{2\,g\,h}}{\sqrt{1 - \left(\frac{A}{A_e}\right)^2}} \tag{7.65}$$

The nature of first order Ordinary Differential Equation that they allow only one initial condition. This initial condition is the initial height of the liquid. The initial velocity field was eliminated by the approximation (remove the acceleration term). Thus it is assumed that the initial velocity is not relevant at the core of the process at hand. It is

[17] For the initial condition speed of sound has to be taken into account. Thus for a very short time, the information about opening of the valve did not reached to the surface. This information travel in characteristic sound speed which is over 1000 m/sec. However, if this phenomenon is ignored this solution is correct.

correct only for large ratio of h/r and the error became very substantial for small value of h/r.

Equation (7.65) integrated to yield

$$\left(1 - \left(\frac{A}{A_e}\right)^2\right) \int_{h_0}^{h} \frac{dh}{\sqrt{2\,g\,h}} = \int_0^t dt \tag{7.66}$$

The initial condition has been inserted into the integral which its solution is

$$\left(1 - \left(\frac{A}{A_e}\right)^2\right) \frac{h - h_0}{\sqrt{2\,g\,h}} = t \tag{7.67}$$

$$U_e = \frac{dh}{dt}\frac{A}{A_e} = \frac{\sqrt{2\,g\,h}}{\sqrt{1 - \left(\frac{A}{A_e}\right)^2}}\frac{A}{A_e} = \frac{\sqrt{2\,g\,h}}{\sqrt{1 - \left(\frac{A_e}{A}\right)^2}} \tag{7.68}$$

If the area ratio $A_e/A << 1$ then

$$U \cong \sqrt{2\,g\,h} \tag{7.69}$$

Equation (7.69) is referred in the literature as Torricelli's equation[18]

This analysis has several drawbacks which limits the accuracy of the calculations. Yet, this analysis demonstrates the usefulness of the integral analysis to provide a reasonable solution. This analysis can be improved by experimental investigating the phenomenon. The experimental coefficient can be added to account for the dissipation and other effects such

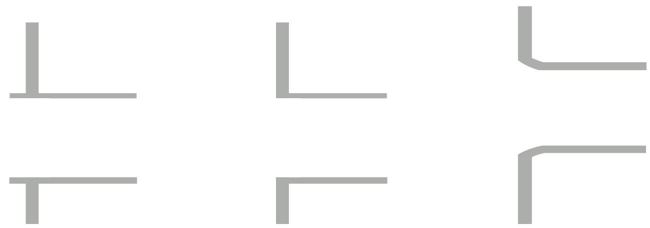
$$\frac{dh}{dt} \cong C\,\sqrt{2\,g\,h} \tag{7.70}$$

The loss coefficient can be expressed as

$$C = K f\left(\frac{U^2}{2}\right) \tag{7.71}$$

A few loss coefficients for different configuration is given following Figure 7.4.

[18]Evangelista Torricelli (October 15, 1608 October 25, 1647) was an Italian physicist and mathematician. He derived this equation based on similar principle to Bernoulli equation (which later leads to Bernoulli's equation). Today the exact reference to his work is lost and only "sketches" of his lecture elude work. He was student (not formal) and follower of Galileo Galilei. It seems that Torricelli was an honest man who gave to others and he died at young age of 39 while in his prime.



(a) Projecting pipe K=1.

(b) Sharp edge pipe connection K=0.5.

(c) Rounded inlet pipe K=0.04.

Fig. -7.4. Typical resistance for selected outlet configuration.

7.2 Limitation of Integral Approach

Some of accuracy issues to enhance the quality and improvements of the integral method were suggested in the analysis of the emptying tank. There are problems that the integral methods even with these enhancements simply cannot tackle.

The improvements to the integral methods are the corrections to the estimates of the energy or other quantities in the conservation equations. In the calculations of the exit velocity of a tank, two such corrections were presented. The first type is the prediction of the velocities profile (or the concentration profile). The second type of corrections is the understanding that averaged of the total field is different from the averaged of different zooms. In the case of the tank, the averaged velocity in x direction is zero yet the averaged velocity in the two zooms (two halves) is not zero. In fact, the averaged energy in the x direction contributes or effects the energy equation. The accuracy issues that integral methods intrinsically suffers from no ability to exact flow field and thus lost the accuracy as was discussed in the example. The integral method does not handle the problems such as the free surface with reasonable accuracy. Furthermore, the knowledge of whether the flow is laminar or turbulent (later on this issue) has to come from different techniques. Hence the prediction can skew the actual predictions.

In the analysis of the tank it was assumed that the dissipation can be ignored. In cases that dissipation play major role, the integral does not provide a sufficient tool to analyze the issue at hand. For example, the analysis of the oscillating manometer cannot be carried by the integral methods. A liquid in manometer is disturbed from a rest by a distance of H_0. The description $H(t)$ as a function of time requires exact knowledge of the velocity field. Additionally, the integral methods is

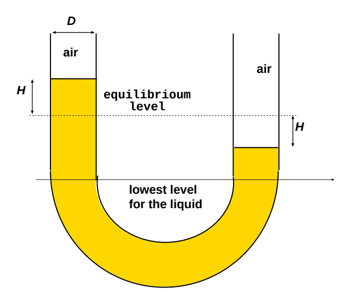


Fig. -7.5. Flow in an oscillating manometer.

too crude to handle issues of free interface.
These problem were minor for the emptying the tank but for the oscillating manometer it is the core of the problem. Hence different techniques are required.

The discussion on the limitations was not provided to discard usage of this method but rather to provide a guidance of use with caution. The integral method is a powerful and yet simple method but has has to be used with the limitations of the method in mind.

7.3 Approximation of Energy Equation

The emptying the tank problem was complicated even with all the simplifications that were carried. Engineers in order to reduce the work further simplify the energy equation. It turn out that these simplifications can provide reasonable results and key understanding of the physical phenomena and yet with less work, the problems can be solved. The following sections provides further explanation.

7.3.1 Energy Equation in Steady State

The steady state situation provides several ways to reduce the complexity. The time derivative term can be eliminated since the time derivative is zero. The acceleration term must be eliminated for the obvious reason. Hence the energy equation is reduced to

Steady State Equation

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \int_S \left(h + \frac{U^2}{2} + g\,z \right) U_{rn}\,\rho\, dA + \int_S P U_{bn} dA \tag{7.72}$$

If the flow is uniform or can be estimated as uniform, equation (7.72) is reduced to

Steady State Equation & uniform

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \left(h + \frac{U^2}{2} + g\,z \right) U_{rn}\,\rho A_{out} - \left(h + \frac{U^2}{2} + g\,z \right) U_{rn}\,\rho A_{in} + P U_{bn} A_{out} - P U_{bn} A_{in} \tag{7.73}$$

It can be noticed that last term in equation (7.73) for non-deformable control volume does not vanished. The reason is that while the velocity is constant, the pressure is different. For a stationary fix control volume the energy equation, under this simplification transformed to

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \left(h + \frac{U^2}{2} + g\,z \right) U_{rn}\,\rho A_{out} - \left(h + \frac{U^2}{2} + g\,z \right) U_{rn}\,\rho A_{in} \tag{7.74}$$

Dividing equation the mass flow rate provides

Steady State Equation, Fix $\dot{m}$ & uniform

$$\dot{q} - \dot{w}_{shear} - \dot{w}_{shaft} = \left. \left(h + \frac{U^2}{2} + g\, z \right) \right|_{out} - \left. \left(h + \frac{U^2}{2} + g\, z \right) \right|_{in} \tag{7.75}$$

7.3.2 Energy Equation in Frictionless Flow and Steady State

In cases where the flow can be estimated without friction or where a quick solution is needed the friction and other losses are illuminated from the calculations. This imaginary fluid reduces the amount of work in the calculations and Ideal Flow Chapter is dedicated in this book. The second low is the core of "no losses" and can be employed when calculations of this sort information is needed. Equation (2.21) which can be written as

$$dq_{rev} = T\, ds = dE_u + P\, dv \tag{7.76}$$

Using the multiplication rule change equation (7.76)

$$dq_{rev} = dE_u + d\,(P\, v) - v\, dP = dE_u + d \left(\frac{P}{\rho} \right) - v\, dP \tag{7.77}$$

integrating equation (7.77) yields

$$\int dq_{rev} = \int dE_u + \int d \left(\frac{P}{\rho} \right) - \int v\, dP \tag{7.78}$$

$$q_{rev} = E_u + \left(\frac{P}{\rho} \right) - \int \frac{dP}{\rho} \tag{7.79}$$

Integration over the entire system results in

$$Q_{rev} = \int_V \overbrace{\left(E_u + \left(\frac{P}{\rho} \right) \right)}^{h} \rho\, dV - \int_V \left(\int \frac{dP}{\rho} \right) \rho\, dV \tag{7.80}$$

Taking time derivative of the equation (7.80) becomes

$$\dot{Q}_{rev} = \frac{D}{Dt} \int_V \overbrace{\left(E_u + \left(\frac{P}{\rho} \right) \right)}^{h} \rho\, dV - \frac{D}{Dt} \int_V \left(\int \frac{dP}{\rho} \right) \rho\, dV \tag{7.81}$$

Using the Reynolds Transport Theorem to transport equation to control volume results in

$$\dot{Q}_{rev} = \frac{d}{dt} \int_V h\, \rho\, dV + \int_A h\, U_{rn}\, \rho\, dA + \frac{D}{Dt} \int_V \left(\int \frac{dP}{\rho} \right) \rho\, dV \tag{7.82}$$

As before equation (7.81) can be simplified for uniform flow as

$$\dot{Q}_{rev} = \dot{m}\left[(h_{out} - h_{in}) - \left(\int \frac{dP}{\rho}\bigg|_{out} - \int \frac{dP}{\rho}\bigg|_{in}\right)\right] \tag{7.83}$$

or

$$\dot{q}_{rev} = (h_{out} - h_{in}) - \left(\int \frac{dP}{\rho}\bigg|_{out} - \int \frac{dP}{\rho}\bigg|_{in}\right) \tag{7.84}$$

Subtracting equation (7.84) from equation (7.75) results in

$$0 = w_{shaft} + \overbrace{\left(\int \frac{dP}{\rho}\bigg|_{2} - \int \frac{dP}{\rho}\bigg|_{1}\right)}^{\text{change in pressure energy}} + \overbrace{\frac{{U_2}^2 - {U_1}^2}{2}}^{\text{change in kinetic energy}} + \overbrace{g\left(z_2 - z_1\right)}^{\text{change in potential energy}} \tag{7.85}$$

Equation (7.85) for constant density is

$$0 = w_{shaft} + \frac{P_2 - P_1}{\rho} + \frac{{U_2}^2 - {U_1}^2}{2} + g\left(z_2 - z_1\right) \tag{7.86}$$

For no shaft work equation (7.86) reduced to

$$0 = \frac{P_2 - P_1}{\rho} + \frac{{U_2}^2 - {U_1}^2}{2} + g\left(z_2 - z_1\right) \tag{7.87}$$

7.4 Energy Equation in Accelerated System

In the discussion so far, it was assumed that the control volume is at rest. The only acceptation to the above statement, is the gravity that was compensated by the gravity potential. In building the gravity potential it was assumed that the gravity is a conservative force. It was pointed earlier in this book that accelerated forces can be translated to potential force. In many cases, the control volume is moving in accelerated coordinates. These accelerations will be translated to potential energy.

The accelerations are referring to two kinds of acceleration, linear and rotational. There is no conceptional difference between these two accelerations. However, the mathematical treatment is somewhat different which is the reason for the separation. General Acceleration can be broken into a linear acceleration and a rotating acceleration.

7.4.1 Energy in Linear Acceleration Coordinate

The potential is defined as

$$P.E. = -\int_{ref}^{2} \boldsymbol{F} \cdot \boldsymbol{d\ell} \tag{7.88}$$

In Chapter 3 a discussion about gravitational energy potential was presented. For example, for the gravity force is

$$F = -\frac{G\,M\,m}{r^2} \tag{7.89}$$

Where G is the gravity coefficient and M is the mass of the Earth. r and m are the distance and mass respectively. The gravity potential is then

$$PE_{gravity} = -\int_{\infty}^{r} -\frac{G\,M\,m}{r^2} dr \tag{7.90}$$

The reference was set to infinity. The gravity force for fluid element in small distance then is $g\,dz\,dm$. The work this element moving from point 1 to point 2 is

$$\int_1^2 g\,dz\,dm = g\,\left(z_2 - z_1\right) dm \tag{7.91}$$

The total work or potential is the integral over the whole mass.

7.4.2 Linear Accelerated System

The acceleration can be employed in similar fashion as the gravity force. The linear acceleration "creates" a conservative force of constant force and direction. The "potential" of moving the mass in the field provides the energy. The Force due to the acceleration of the field can be broken into three coordinates. Thus, the element of the potential is

$$d\,PE_a = \boldsymbol{a} \cdot d\boldsymbol{\ell}\,dm \tag{7.92}$$

The total potential for element material

$$PE_a = \int_{(0)}^{(1)} \boldsymbol{a} \cdot d\boldsymbol{\ell}\,dm = \left(a_x\,(x_1 - x_0)\,a_y\,(y_1 - y_0)\,a_z\,(z_1 - z_0)\right)\,dm \tag{7.93}$$

At the origin (of the coordinates) $x = 0$, $y = 0$, and $z = 0$. Using this trick the notion of the $a_x\,(x_1 - x_0)$ can be replaced by $a_x\,x$. The same can be done for the other two coordinates. The potential of unit material is

$$PE_{a total} = \int_{sys} \left(a_x\,x + a_y\,y + a_z\,z\right)\,\rho\,dV \tag{7.94}$$

The change of the potential with time is

$$\frac{D}{Dt} PE_{a total} = \frac{D}{Dt} \int_{sys} \left(a_x\,x + a_y\,y + a_z\,z\right)\,dm \tag{7.95}$$

Equation can be added to the energy equation as

$$\dot{Q} - \dot{W} = \frac{D}{Dt}\int_{sys} \left[E_u + \frac{U^2}{2} + a_x\,x + a_y\,y + (a_z + g)z\right]\rho\,dV \tag{7.96}$$

The Reynolds Transport Theorem is used to transferred the calculations to control volume as

Energy Equation in Linear Accelerated Coordinate

$$\begin{aligned}\dot{Q} - \dot{W} = \frac{d}{dt}\int_{cv} &\left[E_u + \frac{U^2}{2} + a_x\,x + a_y\,y + (a_z + g)z\right]\rho\,dV \\ &+ \int_{cv}\left(h + \frac{U^2}{2} + a_x\,x + a_y\,y + (a_z + g)z\right)U_{rn}\,\rho\,dA \\ &+ \int_{cv} P\,U_{bn}\,dA\end{aligned} \tag{7.97}$$

7.4.3 Energy Equation in Rotating Coordinate System

The coordinate system rotating around fix axises creates a similar conservative potential as a linear system. There are two kinds of acceleration due to this rotation; one is the centrifugal and one the Coriolis force. To understand it better, consider a particle which moves with the our rotating system. The forces acting on particles are

$$\boldsymbol{F} = \left(\overbrace{\omega^2\,r\,\hat{r}}^{centrifugal} + \overbrace{2\boldsymbol{U}\times\boldsymbol{\omega}}^{Coriolis}\right)dm \tag{7.98}$$

The work or the potential then is

$$PE = \left(\omega^2\,r\,\hat{r} + 2\boldsymbol{U}\times\boldsymbol{\omega}\right)\cdot d\boldsymbol{\ell}\,dm \tag{7.99}$$

The cylindrical coordinate are

$$d\boldsymbol{\ell} = dr\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{k} \tag{7.100}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{k}$ are units vector in the coordinates r, θ and z respectively. The potential is then

$$PE = \left(\omega^2\,r\,\hat{r} + 2\boldsymbol{U}\times\boldsymbol{\omega}\right)\cdot\left(dr\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{k}\right)dm \tag{7.101}$$

The first term results in $\omega^2\,r^2$ (see for explanation in the appendix 363 for vector explanation). The cross product is zero of

$$\boldsymbol{U}\times\boldsymbol{\omega}\times\boldsymbol{U} = \boldsymbol{U}\times\boldsymbol{\omega}\times\boldsymbol{\omega} = 0$$

because the first multiplication is perpendicular to the last multiplication. The second part is

$$(2\boldsymbol{U} \times \boldsymbol{\omega}) \cdot d\boldsymbol{\ell}\, dm \tag{7.102}$$

This multiplication does not vanish with the exception of the direction of $\boldsymbol{U}$. However, the most important direction is the direction of the velocity. This multiplication creates lines (surfaces) of constant values. From a physical point of view, the flux of this property is important only in the direction of the velocity. Hence, this term canceled and does not contribute to the potential.

The net change of the potential energy due to the centrifugal motion is

$$PE_{centrifugal} = -\int_1^2 \omega^2\, r^2\, dr\, dm = \frac{\omega^2 \left({r_1}^2 - {r_2}^2\right)}{2}\, dm \tag{7.103}$$

Inserting the potential energy due to the centrifugal forces into the energy equation yields

Energy Equation in Accelerated Coordinate

$$\begin{aligned}\dot{Q} - \dot{W} = \frac{d}{dt}\int_{cv} \left[E_u + \frac{U^2}{2} + a_x\, x + a_y\, y + (a_z + g) z - \frac{\omega^2\, r^2}{2}\right] \rho\, dV \\ + \int_{cv} \left(h + \frac{U^2}{2} + a_x\, x + a_y\, y + (a_z + g) - z\, \frac{\omega^2\, r^2}{2}\right) U_{rn}\, \rho\, dA \\ + \int_{cv} P\, U_{bn}\, dA\end{aligned} \tag{7.104}$$

7.4.4 Simplified Energy Equation in Accelerated Coordinate

7.4.4.1 Energy Equation in Accelerated Coordinate with Uniform Flow

One of the way to simplify the general equation (7.104) is to assume uniform flow. In that case the time derivative term vanishes and equation (7.104) can be written as

Energy Equation in steady state

$$\begin{aligned}\dot{Q} - \dot{W} = \int_{cv} \left(h + \frac{U^2}{2} + a_x\, x + a_y\, y + (a_z + g) - z\, \frac{\omega^2\, r^2}{2}\right) U_{rn}\, \rho\, dA \\ + \int_{cv} P\, U_{bn}\, dA\end{aligned} \tag{7.105}$$

Further simplification of equation (7.105) by assuming uniform flow for which

$$\dot{Q} - \dot{W} = \left(h + \frac{\overline{U}^2}{2} + a_x\, x + a_y\, y + (a_z + g) - z\, \frac{\omega^2\, r^2}{2} \right) \overline{U}_{rn}\, \rho\, dA + \int_{cv} P\, \overline{U}_{bn}\, dA \tag{7.106}$$

Note that the acceleration also have to be averaged. The correction factors have to introduced into the equation to account for the energy averaged verse to averaged velocity (mass averaged). These factor make this equation with larger error and thus less effective tool in the engineering calculation.

7.4.5 Energy Losses in Incompressible Flow

In the previous sections discussion, it was assumed that there are no energy loss. However, these losses are very important for many real world application. And these losses have practical importance and have to be considered in engineering system. Hence writing equation (7.15) when the energy and the internal energy as a separate identity as

$$\begin{aligned} \dot{W}_{shaft} = \frac{d}{dt} \int_V \left(\frac{U^2}{2} + g\, z \right) \rho\, dV + \\ \int_A \left(\frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) U_{rn}\, \rho\, dA + \int_A P U_{bn} dA + \\ \overbrace{\frac{d}{dt} \int_V E_u\, \rho\, dV + \int_A E_u\, U_{rn}\, \rho\, dA - \dot{Q} - \dot{W}_{shear}}^{\text{energy loss}} \end{aligned} \tag{7.107}$$

Equation (7.107) sometimes written as

$$\begin{aligned} \dot{W}_{shaft} = \frac{d}{dt} \int_V \left(\frac{U^2}{2} + g\, z \right) \rho\, dV + \\ \int_A \left(\frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) U_{rn}\, \rho\, dA + \int_A P U_{bn} dA + \text{energy loss} \end{aligned} \tag{7.108}$$

Equation can be further simplified under assumption of uniform flow and steady state as

$$\dot{w}_{shaft} = \left(\frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) \bigg|_{out} - \left(\frac{P}{\rho} + \frac{U^2}{2} + g\, z \right) \bigg|_{in} + \text{energy loss} \tag{7.109}$$

Equation (7.109) suggests that term $h + \frac{U^2}{2} + g\, z$ has a special meaning (because it remained constant under certain conditions). This term, as will be shown, has to be constant for frictionless flow without any addition and loss of energy. This term represents the "potential energy." The loss is the combination of the internal energy/enthalpy with heat transfer. For example, fluid flow in a pipe has resistance and energy dissipation. The dissipation is lost energy that is transferred to the surroundings. The loss is normally is a strong function of the velocity square, $U^2/2$. There are several categories of the loss which referred as minor loss (which are not minor), and duct losses. These losses will be tabulated later on.

If the energy loss is negligible and the shaft work vanished or does not exist equation (7.109) reduces to simple Bernoulli's equation.

Simple Bernoulli

$$0 = \left(\frac{P}{\rho} + \frac{U^2}{2} + g\,z\right)\bigg|_{out} - \left(\frac{P}{\rho} + \frac{U^2}{2} + g\,z\right)\bigg|_{in} \tag{7.110}$$

Equation (7.110) is only a simple form of Bernoulli's equation which was developed by Bernoulli's adviser, Euler. There also unsteady state and other form of this equation that will be discussed in differential equations Chapter.

7.5 Examples of Integral Energy Conservation

Example 7.1:

Consider a flow in a long straight pipe. Initially the flow is in a rest. At time, t_0 the a constant pressure difference is applied on the pipe. Assume that flow is incompressible, and the resistance or energy loss is f. Furthermore assume that this loss is a function of the velocity square. Develop equation to describe the exit velocity as a function of time. State your assumptions.

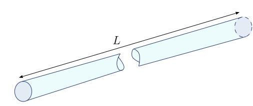


Fig. -7.6. Flow in a long pipe when exposed to a jump in the pressure difference.

SOLUTION

The mass balance on the liquid in the pipe results in

$$0 = \overbrace{\int_V \frac{\partial \rho}{\partial t} dV}^{=0} + \overbrace{\int_A \rho\, U_{bn} dA}^{=0} + \int_A \rho\, U_{rn} dA \Longrightarrow \not\rho \not A\, U_{in} = \not\rho \not A\, U_{exit} \tag{7.I.a}$$

There is no change in the liquid mass inside pipe and therefore the time derivative is zero (the same mass resides in the pipe at all time). The boundaries do not move and the second term is zero. Thus, the flow in and out are equal because the density is identical. Furthermore, the velocity is identical because the cross area is same.

It can be noticed that for the energy balance on the pipe, the time derivative can enter the integral because the control volume has fixed boundaries. Hence,

$$\dot{Q} - \overbrace{\dot{W}_{shear}}^{=0} + \overbrace{\dot{W}_{shaft}}^{=0} = \int_V \frac{d}{dt}\left(E_u + \frac{U^2}{2} + g\,z\right)\rho\, dV + \int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\, dA + \int_S P U_{bn} dA \tag{7.I.b}$$

The boundaries shear work vanishes because the same arguments present before (the work, where velocity is zero, is zero. In the locations where the velocity does not vanished, such as in and out, the work is zero because shear stress are perpendicular to the velocity).

There is no shaft work and this term vanishes as well. The first term on the right hand side (with a constant density) is

$$\rho \int_{V_{pipe}} \frac{d}{dt}\left(E_u + \frac{U^2}{2} + \overbrace{g\,z}^{constant}\right) dV = \rho\, U\, \frac{d\,U}{dt}\, \overbrace{V_{pipe}}^{L\,\pi\,r^2} + \rho \int_{V_{pipe}} \frac{d}{dt}\left(E_u\right) dV \tag{7.I.c}$$

where L is the pipe length, r is the pipe radius, U averaged velocity.

In this analysis, it is assumed that the pipe is perpendicular to the gravity line and thus the gravity is constant. The gravity in the first term and all other terms, related to the pipe, vanish again because the value of z is constant. Also, as can be noticed from equation (7.I.a), the velocity is identical (in and out). Hence the second term becomes

$$\int_A \left(h + \left(\cancel{\frac{U^2}{2} + g\,z}\right)^{constant}\right) \rho\, U_{rn} dA = \int_A \overbrace{\left(E_u + \frac{P}{\rho}\right)}^{h} \rho\, U_{rn} dA \tag{7.I.d}$$

Equation (7.I.d) can be further simplified (since the area and averaged velocity are constant, additionally notice that $U = U_{rn}$) as

$$\int_A \left(E_u + \frac{P}{\rho}\right) \rho\, U_{rn} dA = \Delta P\, U\, A + \int_A \rho\, E_u\, U_{rn}\, dA \tag{7.I.e}$$

The third term vanishes because the boundaries velocities are zero and therefore

$$\int_A P\; U_{bn} dA = 0 \tag{7.I.f}$$

Combining all the terms results in

$$\dot{Q} = \rho\, U\, \frac{d\,U}{dt}\, \overbrace{V_{pipe}}^{L\,\pi\,r^2} + \rho\, \frac{d}{dt} \int_{V_{pipe}} E_u\, dV + \Delta P\, U\, dA + \int_A \rho\, E_u\, U\, dA \tag{7.I.g}$$

equation (7.I.g) can be rearranged as

$$\overbrace{\dot{Q} - \rho \int_{V_{pipe}} \frac{d\left(E_u\right)}{dt}\, dV - \int_A \rho\, E_u\, U\, dA}^{-K\frac{U^2}{2}} = \rho\, L\, \pi\, r^2\, U\, \frac{d\,U}{dt} + \left(P_{in} - P_{out}\right) U \tag{7.I.h}$$

The terms on the LHS (left hand side) can be combined. It common to assume (to view) that these terms are representing the energy loss and are a strong function of velocity square[19]. Thus, equation (7.I.h) can be written as

$$-K \frac{U^2}{2} = \rho\, L\, \pi\, r^2\, U \frac{dU}{dt} + (P_{in} - P_{out})\, U \tag{7.I.i}$$

Dividing equation (7.I.i) by $K\,U/2$ transforms equation (7.I.i) to

$$U + \frac{2\,\rho\, L\, \pi\, r^2}{K} \frac{dU}{dt} = \frac{2\,(P_{in} - P_{out})}{K} \tag{7.I.j}$$

Equation (7.I.j) is a first order differential equation. The solution this equation is described in the appendix and which is

$$U = e^{-\left(\frac{t\,K}{2\,\pi\, r^2\, \rho\, L}\right)} \left(\frac{2\,(P_{in} - P_{out})\, e^{\left(\frac{t\,K}{2\,\pi\, r^2\, \rho\, L}\right)}}{K} + c \right) e^{\left(\frac{2\,\pi\, r^2\, \rho\, t\, L}{K}\right)} \tag{7.I.k}$$

Applying the initial condition, $U(t = 0) = 0$ results in

$$U = \frac{2\,(P_{in} - P_{out})}{K} \left(1 - e^{-\left(\frac{t\,K}{2\,\pi\, r^2\, \rho\, L}\right)} \right) \tag{7.I.l}$$

The solution is an exponentially approaching the steady state solution. In steady state the flow equation (7.I.j) reduced to a simple linear equation. The solution of the linear equation and the steady state solution of the differential equation are the same.

$$U = \frac{2\,(P_{in} - P_{out})}{K} \tag{7.I.m}$$

Another note, in reality the resistance, K, is not constant but rather a strong function of velocity (and other parameters such as temperature[20], velocity range, velocity regime and etc.). This function will be discussed in a greater extent later on. Additionally, it should be noted that if momentum balance was used a similar solution (but not the same) was obtained (why? hint the difference of the losses accounted for).

End Solution

The following example combined the above discussion in the text with the above example (7.1).

[19]The shear work inside the liquid refers to molecular work (one molecule work on the other molecule). This shear work can be viewed also as one control volume work on the adjoined control volume.

[20]Via the viscosity effects.

Example 7.2:

A large cylindrical tank with a diameter, D, contains liquid to height, h. A long pipe is connected to a tank from which the liquid is emptied. To analysis this situation, consider that the tank has a constant pressure above liquid (actually a better assumption of air with a constant mass.). The pipe is exposed to the surroundings and thus the pressure is P_{atmos} at the pipe exit. Derive approximated equations that related the height in the large tank and the exit velocity at the pipe to pressure difference. Assume that the liquid is incompressible. Assume that the resistance or the friction in the pipe is a strong function to the velocity square in the tank. State all the assumptions that were made during the derivations.

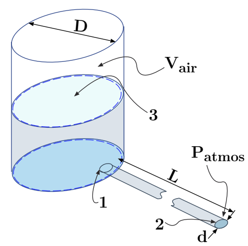


Fig. -7.7. Liquid exiting a large tank trough a long tube.

Solution

This problem can split into two control volumes; one of the liquid in the tank and one of the liquid in pipe. Analysis of control volume in the tank was provided previously and thus needed to be sewed to Example 7.1. Note, the energy loss is considered (as opposed to the discussion in the text). The control volume in tank is depicted in Figure 7.7.

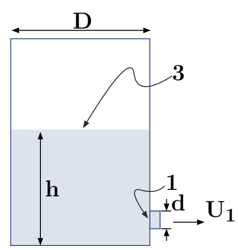


Fig. -7.8. Tank control volume for Example 7.2.

Tank Control Volume

The effect of the energy change in air side was neglected. The effect is negligible in most cases because air mass is small with exception the "spring" effect (expansion/compression effects). The mass conservation reads

$$\overbrace{\int_V \frac{\partial \rho}{\partial t} dV}^{=0} + \int_A \rho\, U_{bn} dA + \int_A \rho\, U_{rn} dA = 0 \tag{7.II.a}$$

The first term vanishes and the second and third terms remain and thus equation (7.II.a) reduces to

$$\not\rho\, U_1\, A_{pipe} = \not\rho\, U_3 \overbrace{\pi\, R^2}^{A_{tank}} = \not\rho\, \frac{dh}{dt} \overbrace{\pi\, R^2}^{A_{tank}} \tag{7.II.b}$$

It can be noticed that $U_3 = dh/dt$ and $D = 2\,R$ and $d = 2\,r$ when the lower case refers to the pipe and the upper case referred to the tank. Equation (7.II.b) simply can be written when the area ratio is used (to be changed later if needed) as

$$U_1\, A_{pipe} = \frac{dh}{dt}\, A_{tank} \Longrightarrow U_1 = \left(\frac{R}{r}\right)^2 \frac{dh}{dt} \tag{7.II.c}$$

The boundaries shear work and the shaft work are assumed to be vanished in the tank. Therefore, the energy conservation in the tank reduces to

$$\begin{aligned} \dot{Q} - \overbrace{\dot{W}_{shear}}^{=0} + \quad \overbrace{\dot{W}_{shaft}}^{=0} = \frac{d}{dt}\int_{V_t}\left(E_u + \frac{{U_t}^2}{2} + g\,z\right)\rho\, dV + \\ \int_{A_1}\left(h + \frac{{U_t}^2}{2} + g\,z\right) U_{rn}\,\rho\, dA + \int_{A_3} P U_{bn} dA \end{aligned} \tag{7.II.d}$$

Where U_t denotes the (the upper surface) liquid velocity of the tank. Moving all internal energy terms and the energy transfer to the right hand side of equation (7.II.d) to become

$$\begin{aligned} \frac{d}{dt}\int_{V_t}\left(\frac{{U_t}^2}{2} + g\,z\right)\rho\, dV + \int_{A_1}\left(\frac{P}{\rho} + \frac{{U_t}^2}{2} + g\,z\right)\overbrace{U_{rn}}^{U_1}\,\rho\, dA + \\ \int_{A_3} P\,\overbrace{U_{bn}}^{U_3}\, dA = \overbrace{\frac{d}{dt}\int_{V_t} E_u \rho\, dV + \int_{A_1} E_u\,\rho\, U_{rn}\, dA - \dot{Q}}^{K\,\frac{{U_t}^2}{2}} \end{aligned} \tag{7.111}$$

Similar arguments to those that were used in the previous discussion are applicable to this case. Using equation (7.38), the first term changes to

$$\frac{d}{dt}\int_{V}\rho\left(\frac{U^2}{2} + g\,z\right) dV \quad \cong \rho\,\frac{d}{dt}\left(\left[\frac{\overline{U_t}^2}{2} + \frac{g\,h}{2}\right]\overbrace{h\,A}^{V}\right) \tag{7.II.e}$$

Where the velocity is given by equation (7.44). That is, the velocity is a derivative of the height with a correction factor, $U = dh/dt \times f(G)$. Since the focus in this book is primarily on the physics, $f(G) \equiv 1$ will be assumed. The pressure component of the second term is

$$\int_A \frac{P}{\not\rho}\, U_{rn}\, \not\rho\, dA = \rho\, P_1\, U_1\, A_1 \tag{7.II.f}$$

It is assumed that the exit velocity can be averaged (neglecting the velocity distribution effects). The second term can be recognized as similar to those by equation (7.45). Hence, the second term is

$$\int_A\left(\frac{U^2}{2} + \overbrace{g\,z}^{z=0}\right) U_{rn}\,\rho\, dA \cong \frac{1}{2}\left(\frac{dh}{dt}\frac{A_3}{A_1}\right)^2 U_1\,\rho\, A_1 = \frac{1}{2}\left(\frac{dh}{dt}\frac{R}{r}\right)^2 U_1\,\rho\, A_1 \tag{7.II.g}$$

The last term on the left hand side is

$$\int_A P U_{bn} dA = P_3 \, A \, \frac{dh}{dt} \tag{7.II.h}$$

The combination of all the terms for the tank results in

$$\frac{d}{dt}\left(\left[\frac{\overline{U_t}^2}{2} + \frac{g\,h}{2}\right] \overbrace{h\,A}^{V}\right) - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A_3}{A_1}\right)^2 U_1 \; A_1 + \frac{K_t}{2\,\rho}\left(\frac{dh}{dt}\right)^2 = \frac{(P_3 - P_1)}{\rho} \tag{7.II.i}$$

Pipe Control Volume

The analysis of the liquid in the pipe is similar to Example 7.1. The conservation of the liquid in the pipe is the same as in Example 7.1 and thus equation (7.I.a) is used

$$U_1 = U_2 \tag{7.II.j}$$

$$U_p + \frac{4\,\rho\,L\,\pi\,r^2}{K_p}\frac{d\,U_p}{dt} = \frac{2\,(P_1 - P_2)}{K_p} \tag{7.II.k}$$

where K_p is the resistance in the pipe and U_p is the (averaged) velocity in the pipe. Using equation (7.II.c) eliminates the U_p as

$$\frac{dh}{dt} + \frac{4\,\rho\,L\,\pi\,r^2}{K}\frac{d^2\,h}{dt^2} = \left(\frac{R}{r}\right)^2 \frac{2\,(P_1 - P_2)}{K_p} \tag{7.II.l}$$

Equation (7.II.l) can be rearranged as

$$\frac{K_p}{2\,\rho}\left(\frac{r}{R}\right)^2 \left(\frac{dh}{dt} + \frac{4\,\rho\,L\,\pi\,r^2}{K}\frac{d^2\,h}{dt^2}\right) = \frac{(P_1 - P_2)}{\rho} \tag{7.II.m}$$

Solution

The equations (7.II.m) and (7.II.i) provide the frame in which the liquid velocity in tank and pipe have to be solved. In fact, it can be noticed that the liquid velocity in the tank is related to the height and the liquid velocity in the pipe. Thus, there is only one equation with one unknown. The relationship between the height was obtained by substituting equation (7.II.c) in equation (7.II.m). The equations (7.II.m) and (7.II.i) have two unknowns (dh/dt and P_1) which are sufficient to solve the problem. It can be noticed that two initial conditions are required to solve the problem.

The governing equation obtained by from adding equation (7.II.m) and (7.II.i) as

$$\begin{aligned} \frac{d}{dt}\left(\left[\frac{\overline{U_t}^2}{2} + \frac{g\,h}{2}\right] \overbrace{h\,A}^{V}\right) - \frac{1}{2}\left(\frac{dh}{dt}\right)^2 \left(\frac{A_3}{A_1}\right)^2 U_1 \; A_1 + \frac{K_t}{2\,\rho}\left(\frac{dh}{dt}\right)^2 \\ + \frac{K_p}{2\,\rho}\left(\frac{r}{R}\right)^2 \left(\frac{dh}{dt} + \frac{4\,\rho\,L\,\pi\,r^2}{K}\frac{d^2\,h}{dt^2}\right) = \frac{(P_3 - P_2)}{\rho} \end{aligned} \tag{7.II.n}$$

The initial conditions are that zero initial velocity in the tank and pipe. Additionally, the height of liquid is at prescript point as

$$\begin{aligned} h(0) &= h_0 \\ \frac{dh}{dt}(0) &= 0 \end{aligned} \tag{7.II.o}$$

The solution of equation can be obtained using several different numerical techniques. The dimensional analysis method can be used to obtain solution various situations which will be presented later on.

End Solution

Qualitative Questions

- A liquid flows in and out from a long pipe with uniform cross section as single phase. Assume that the liquid is slightly compressible. That is the liquid has a constant bulk modulus, B_T. What is the direction of the heat from the pipe or in to the pipe. Explain why the direction based on physical reasoning. What kind of internal work the liquid performed. Would happen when the liquid velocity is very large? What it will be still correct.

- A different liquid flows in the same pipe. If the liquid is compressible what is the direction of the heat to keep the flow isothermal?

- A tank is full of incompressible liquid. A certain point the tank is punctured and the liquid flows out. To keep the tank at uniform temperature what is the direction of the heat (from the tank or to the tank)?