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CHAPTER 2: FLUID INTRO

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This chapter is part of the textbook:

“Basics of Fluid Mechanics”

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GENICK BAR-MEIR, PH.D.
CHICAGO, ILLINOIS
DECEMBER 21, 2011

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NSY = Not Started Yet

CHAPTER 1

Introduction to Fluid Mechanics

1.1 What is Fluid Mechanics?

Fluid mechanics deals with the study of all fluids under static and dynamic situations. Fluid mechanics is a branch of continuous mechanics which deals with a relationship between forces, motions, and static conditions in a continuous material. This study area deals with many and diversified problems such as surface tension, fluid statics, flow in enclosed bodies, or flow around bodies (solid or otherwise), flow stability, etc. In fact, almost any action a person is doing involves some kind of a fluid mechanics problem. Furthermore, the boundary between the solid mechanics and fluid mechanics is some kind of gray shed and not a sharp distinction (see Figure 1.1 for the complex relationships between the different branches which only part of it should be drawn in the same time.). For example, glass appears as a solid material, but a closer look reveals that the glass is a liquid with a large viscosity. A proof of the glass “liquidity” is the change of the glass thickness in high windows in European Churches after hundred years. The bottom part of the glass is thicker than the top part. Materials like sand (some call it quick sand) and grains should be treated as liquids. It is known that these materials have the ability to drown people. Even material such as aluminum just below the mushy zone¹ also behaves as a liquid similarly to butter. Furthermore, material particles that “behaves” as solid mixed with liquid creates a mixture that behaves as a complex² liquid. After it was established that the boundaries of fluid mechanics aren’t sharp, most of the discussion in this book is limited to simple and (mostly) Newtonian (sometimes power fluids) fluids which will be defined later.

The fluid mechanics study involve many fields that have no clear boundaries between them. Researchers distinguish between orderly flow and chaotic flow as the

¹Mushy zone zone refers to to aluminum alloy with partially solid and partially liquid phases.

²It can be viewed as liquid solid multiphase flow.

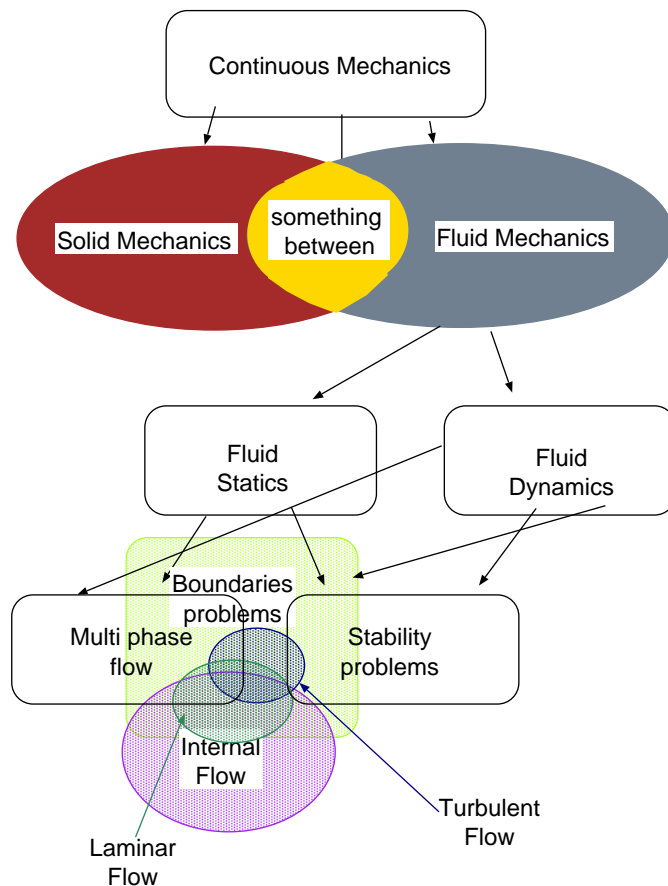


Fig. -1.1. Diagram to explain part of relationships of fluid mechanics branches.

laminar flow and the turbulent flow. The fluid mechanics can also be distinguish between a single phase flow and multiphase flow (flow made more than one phase or single distinguishable material). The last boundary (as all the boundaries in fluid mechanics) isn't sharp because fluid can go through a phase change (condensation or evaporation) in the middle or during the flow and switch from a single phase flow to a multi phase flow. Moreover, flow with two phases (or materials) can be treated as a single phase (for example, air with dust particle).

After it was made clear that the boundaries of fluid mechanics aren't sharp, the study must make arbitrary boundaries between fields. Then the dimensional analysis can be used explain why in certain cases one distinguish area/principle is more relevant than the other and some effects can be neglected. Or, when a general model is need because more parameters are effecting the situation. It is this author's per-

sonal experience that the knowledge and ability to know in what area the situation lay is one of the main problems. For example, engineers in software company (EKK Inc, <http://ekkin.com/HTML>) analyzed a flow of a complete still liquid assuming a complex turbulent flow model. Such absurd analysis are common among engineers who do not know which model can be applied. Thus, one of the main goals of this book is to explain what model should be applied. Before dealing with the boundaries, the simplified private cases must be explained.

There are two main approaches of presenting an introduction of fluid mechanics materials. The first approach introduces the fluid kinematic and then the basic governing equations, to be followed by stability, turbulence, boundary layer and internal and external flow. The second approach deals with the Integral Analysis to be followed with Differential Analysis, and continue with Empirical Analysis. These two approaches pose a dilemma to anyone who writes an introductory book for the fluid mechanics. These two approaches have justifications and positive points. Reviewing many books on fluid mechanics made it clear, there isn't a clear winner. This book attempts to find a hybrid approach in which the kinematic is presented first (aside to standard initial four chapters) follow by Integral analysis and continued by Differential analysis. The ideal flow (frictionless flow) should be expanded compared to the regular treatment. This book is unique in providing chapter on multiphase flow. Naturally, chapters on open channel flow (as a sub class of the multiphase flow) and compressible flow (with the latest developments) are provided.

1.2 *Brief History*

The need to have some understanding of fluid mechanics started with the need to obtain water supply. For example, people realized that wells have to be dug and crude pumping devices need to be constructed. Later, a large population created a need to solve waste (sewage) and some basic understanding was created. At some point, people realized that water can be used to move things and provide power. When cities increased to a larger size, aqueducts were constructed. These aqueducts reached their greatest size and grandeur in those of the City of Rome and China.

Yet, almost all knowledge of the ancients can be summarized as application of instincts, with the exception Archimedes (250 B.C.) on the principles of buoyancy. For example, larger tunnels built for a larger water supply, etc. There were no calculations even with the great need for water supply and transportation. The first progress in fluid mechanics was made by Leonardo Da Vinci (1452-1519) who built the first chambered canal lock near Milan. He also made several attempts to study the flight (birds) and developed some concepts on the origin of the forces. After his initial work, the knowledge of fluid mechanics (hydraulic) increasingly gained speed by the contributions of Galileo, Torricelli, Euler, Newton, Bernoulli family, and D'Alembert. At that stage theory and experiments had some discrepancy. This fact was acknowledged by D'Alembert who stated that, "The theory of fluids must necessarily be based upon experiment." For example the concept of ideal liquid that leads to motion with no resistance, conflicts with the reality.

This discrepancy between theory and practice is called the “D’Alembert paradox” and serves to demonstrate the limitations of theory alone in solving fluid problems. As in thermodynamics, two different schools of thought were created: the first believed that the solution will come from theoretical aspect alone, and the second believed that solution is the pure practical (experimental) aspect of fluid mechanics. On the theoretical side, considerable contributions were made by Euler, La Grange, Helmholtz, Kirchhoff, Rayleigh, Rankine, and Kelvin. On the “experimental” side, mainly in pipes and open channels area, were Brahm, Bossut, Chezy, Dubuat, Fabre, Coulomb, Dupuit, d’Aubisson, Hagen, and Poiseuille.

In the middle of the nineteenth century, first Navier in the molecular level and later Stokes from continuous point of view succeeded in creating governing equations for real fluid motion. Thus, creating a matching between the two schools of thought: experimental and theoretical. But, as in thermodynamics, people cannot relinquish control. As a result it created today “strange” names: Hydrodynamics, Hydraulics, Gas Dynamics, and Aeronautics.

The Navier-Stokes equations, which describes the flow (or even Euler equations), were considered unsolvable during the mid nineteenth century because of the high complexity. This problem led to two consequences. Theoreticians tried to simplify the equations and arrive at approximated solutions representing specific cases. Examples of such work are Hermann von Helmholtz’s concept of vortices (1858), Lanchester’s concept of circulatory flow (1894), and the Kutta-Joukowski circulation theory of lift (1906). The experimentalists, at the same time proposed many correlations to many fluid mechanics problems, for example, resistance by Darcy, Weisbach, Fanning, Ganguillet, and Manning. The obvious happened without theoretical guidance, the empirical formulas generated by fitting curves to experimental data (even sometime merely presenting the results in tabular form) resulting in formulas that the relationship between the physics and properties made very little sense.

At the end of the twenty century, the demand for vigorous scientific knowledge that can be applied to various liquids as opposed to formula for every fluid was created by the expansion of many industries. This demand coupled with new several novel concepts like the theoretical and experimental researches of Reynolds, the development of dimensional analysis by Rayleigh, and Froude’s idea of the use of models change the science of the fluid mechanics. Perhaps the most radical concept that effects the fluid mechanics is of Prandtl’s idea of boundary layer which is a combination of the modeling and dimensional analysis that leads to modern fluid mechanics. Therefore, many call Prandtl as the father of modern fluid mechanics. This concept leads to mathematical basis for many approximations. Thus, Prandtl and his students Blasius, von Karman, Meyer, and Blasius and several other individuals as Nikuradse, Rose, Taylor, Bhuckingham, Stanton, and many others, transformed the fluid mechanics to today modern science.

While the understanding of the fundamentals did not change much, after World War Two, the way how it was calculated changed. The introduction of the computers during the 60s and much more powerful personal computer has changed the field. There are many open source programs that can analyze many fluid mechanics situations. To-

day many problems can be analyzed by using the numerical tools and provide reasonable results. These programs in many cases can capture all the appropriate parameters and adequately provide a reasonable description of the physics. However, there are many other cases that numerical analysis cannot provide any meaningful result (trends). For example, no weather prediction program can produce good engineering quality results (where the snow will fall within 50 kilometers accuracy. Building a car with this accuracy is a disaster). In the best scenario, these programs are as good as the input provided. Thus, assuming turbulent flow for still flow simply provides erroneous results (see for example, EKK, Inc).

1.3 Kinds of Fluids

Some differentiate fluid from solid by the reaction to shear stress. It is a known fact said that the fluid continuously and permanently deformed under shear stress while solid exhibits a finite deformation which does not change with time. It is also said that liquid cannot return to their original state after the deformation. This differentiation leads to three groups of materials: solids and liquids. This test creates a new material group that shows dual behaviors; under certain limits; it behaves like solid and under others it behaves like liquid (see Figure 1.1). The study of this kind of material called rheology and it will (almost) not be discussed in this book. It is evident from this discussion that when a liquid is at rest, no shear stress is applied.

The fluid is mainly divided into two categories: liquids and gases. The main difference between the liquids and gases state is that gas will occupy the whole volume while liquids has an almost fix volume. This difference can be, for most practical purposes considered, sharp even though in reality this difference isn't sharp. The difference between a gas phase to a liquid phase above the critical point are practically minor. But below the critical point, the change of water pressure by 1000% only change the volume by less than 1 percent. For example, a change in the volume by more 5% will required tens of thousands percent change of the pressure. So, if the change of pressure is significantly less than that, then the change of volume is at best 5%. Hence, the pressure will not affect the volume. In gaseous phase, any change in pressure directly affects the volume. The gas fills the volume and liquid cannot. Gas has no free interface/surface (since it does fill the entire volume).

There are several quantities that have to be addressed in this discussion. The first is **force** which was reviewed in physics. The unit used to measure is [N]. It must be remember that force is a vector, e.g it has a direction. The second quantity discussed here is the area. This quantity was discussed in physics class but here it has an additional meaning, and it is referred to the direction of the area. The direction of area is perpendicular to the area. The area is measured in [m^2]. Area of three-dimensional object has no single direction. Thus, these kinds of areas should be addressed infinitesimally and locally.

The traditional quantity, which is force per area has a new meaning. This is a result of division of a vector by a vector and it is referred to as tensor. In this book, the emphasis is on the physics, so at this stage the tensor will have to be broken

into its components. Later, the discussion on the mathematical meaning is presented (later version). For the discussion here, the pressure has three components, one in the area direction and two perpendicular to the area. The pressure component in the area direction is called pressure (great way to confuse, isn't it?). The other two components are referred as the shear stresses. The units used for the pressure components is $[N/m^2]$.

The density is a property which requires that liquid to be continuous. The density can be changed and it is a function of time and space (location) but must have a continuous property. It doesn't mean that a sharp and abrupt change in the density cannot occur. It referred to the fact that density is independent of the sampling size. Figure 1.2 shows the density as a function of the sample size. After certain sample size, the density remains constant. Thus, the density is defined as

$$\rho = \lim_{\Delta V \rightarrow \epsilon} \frac{\Delta m}{\Delta V} \quad (1.1)$$

It must be noted that ϵ is chosen so that the continuous assumption is not broken, that is, it did not reach/reduced to the size where the atoms or molecular statistical calculations are significant (see Figure 1.2 for point where the green lines converge to constant density). When this assumption is broken, then, the principles of statistical mechanics must be utilized.

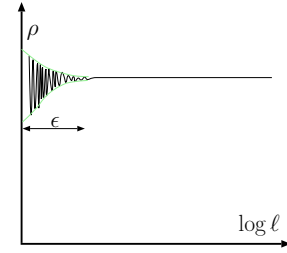


Fig. -1.2. Density as a function of the size of sample.

1.4 Shear Stress

The shear stress is part of the pressure tensor. However, here, and many parts of the book, it will be treated as a separate issue. In solid mechanics, the shear stress is considered as the ratio of the force acting on area in the direction of the forces perpendicular to area. Different from solid, fluid cannot pull directly but through a solid surface. Consider liquid that undergoes a shear stress between a short distance of two plates as shown in Figure (1.3).

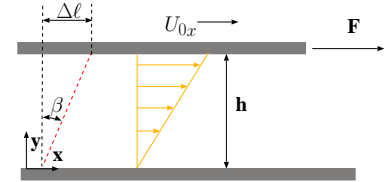


Fig. -1.3. Schematics to describe the shear stress in fluid mechanics.

The upper plate velocity generally will be

$$U = f(A, F, h) \quad (1.2)$$

Where A is the area, the F denotes the force, h is the distance between the plates. From solid mechanics study, it was shown that when the force per area increases, the velocity of the plate increases also. Experiments show that the increase of height will increase the velocity up to a certain range. Consider moving the plate with a zero lubricant ($h \sim 0$) (results in large force) or a large amount of lubricant (smaller force).

In this discussion, the aim is to develop differential equation, thus the small distance analysis is applicable.

For cases where the dependency is linear, the following can be written

$$U \propto \frac{hF}{A} \quad (1.3)$$

Equations (1.3) can be rearranged to be

$$\frac{U}{h} \propto \frac{F}{A} \quad (1.4)$$

Shear stress was defined as

$$\tau_{xy} = \frac{F}{A} \quad (1.5)$$

The index x represent the “direction of the shear stress while the y represent the direction of the area(perpendicular to the area). From equations (1.4) and (1.5) it follows that ratio of the velocity to height is proportional to shear stress. Hence, applying the coefficient to obtain a new equality as

$$\tau_{xy} = \mu \frac{U}{h} \quad (1.6)$$

Where μ is called the absolute viscosity or dynamic viscosity which will be discussed later in this chapter in a great length.

In steady state, the distance the upper plate moves after small amount of time, δt is

$$d\ell = U \delta t \quad (1.7)$$

From Figure 1.4 it can be noticed that for a small angle, $\delta\beta \cong \sin \beta$, the regular approximation provides

$$d\ell = U \delta t = \overbrace{h \delta\beta}^{\text{geometry}} \quad (1.8)$$

From equation (1.8) it follows that

$$U = h \frac{\delta\beta}{\delta t} \quad (1.9)$$

Combining equation (1.9) with equation (1.6) yields

$$\tau_{xy} = \mu \frac{\delta\beta}{\delta t} \quad (1.10)$$

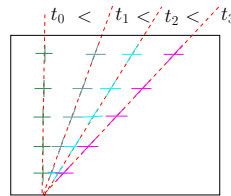


Fig. -1.4. The deformation of fluid due to shear stress as progression of time.

If the velocity profile is linear between the plate (it will be shown later that it is consistent with derivations of velocity), then it can be written for small a angel that

$$\frac{\delta\beta}{\delta t} = \frac{dU}{dy} \quad (1.11)$$

Materials which obey equation (1.10) referred to as Newtonian fluid. For this kind of substance

$$\tau_{xy} = \mu \frac{dU}{dy} \quad (1.12)$$

Newtonian fluids are fluids which the ratio is constant. Many fluids fall into this category such as air, water etc. This approximation is appropriate for many other fluids but only within some ranges.

Equation (1.9) can be interpreted as momentum in the x direction transferred into the y direction. Thus, the viscosity is the resistance to the flow (flux) or the movement. The property of viscosity, which is exhibited by all fluids, is due to the existence of cohesion and interaction between fluid molecules. These cohesion and interactions hamper the flux in y -direction. Some referred to shear stress as viscous flux of x -momentum in the y -direction. The units of shear stress are the same as flux per time as following

$$\frac{F}{A} \left[\frac{kg \ m}{sec^2 \ m^2} \right] = \frac{\dot{m}U}{A} \left[\frac{kg}{sec} \frac{m}{sec} \frac{1}{m^2} \right]$$

Thus, the notation of τ_{xy} is easier to understand and visualize. In fact, this interpretation is more suitable to explain the molecular mechanism of the viscosity. The units of absolute viscosity are $[N \ sec/m^2]$.

Example 1.1:

A space of 1 [cm] width between two large plane surfaces is filled with glycerin. Calculate the force that is required to drag a very thin plate of 1 [m²] at a speed of 0.5 m/sec. It can be assumed that the plates remains in equidistant from each other and steady state is achieved instantly.

SOLUTION

Assuming Newtonian flow, the following can be written (see equation (1.6))

$$F = \frac{A \mu U}{h} \sim \frac{1 \times 1.069 \times 0.5}{0.01} = 53.45[N]$$

End Solution

Example 1.2:

Castor oil at 25°C fills the space between two concentric cylinders of 0.2[m] and 0.1[m] diameters with height of 0.1 [m]. Calculate the torque required to rotate the inner cylinder at 12 rpm, when the outer cylinder remains stationary. Assume steady state conditions.

SOLUTION

The velocity is

$$U = r \dot{\theta} = 2 \pi r_i \text{ rps} = 2 \times \pi \times 0.1 \times \overbrace{12/60}^{\text{rps}} = 0.4 \pi r_i$$

Where *rps* is revolution per second.

The same way as in example (1.1), the moment can be calculated as the force times the distance as

$$M = F \ell = \frac{\overbrace{\ell}^{r_i} \overbrace{A}^{2 \pi r_i h} \mu U}{r_o - r_i}$$

In this case $r_o - r_i = h$ thus,

$$M = \frac{2 \pi^2 \overbrace{0.1^3}^{r_i} \cancel{h} \overbrace{0.986}^{\mu} 0.4}{\cancel{h}} \sim .0078 [N m]$$

End Solution

1.5 Viscosity

1.5.1 General

Viscosity varies widely with temperature. However, temperature variation has an opposite effect on the viscosities of liquids and gases. The difference is due to their fundamentally different mechanism creating viscosity characteristics. In gases, molecules are sparse and cohesion is negligible, while in the liquids, the molecules are more compact and cohesion is more dominant. Thus, in gases, the exchange of momentum between layers brought as a result of

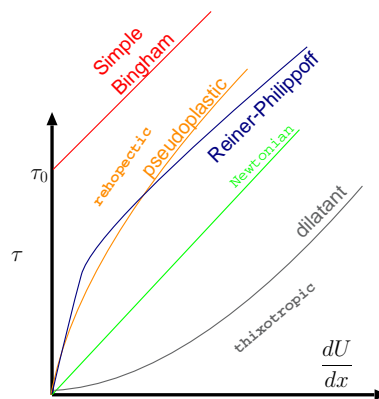


Fig. -1.5. The different of power fluids families.

molecular movement normal to the general direction of flow, and it resists the flow. This molecular activity is known to increase with temperature, thus, the viscosity of gases will increase with temperature. This reasoning is a result of the considerations of the kinetic theory. This theory indicates that gas viscosities vary directly with the square root of temperature. In liquids, the momentum exchange due to molecular movement is small compared to the cohesive forces between the molecules. Thus, the viscosity is

primarily dependent on the magnitude of these cohesive forces. Since these forces decrease rapidly with increases of temperature, liquid viscosities decrease as temperature increases.

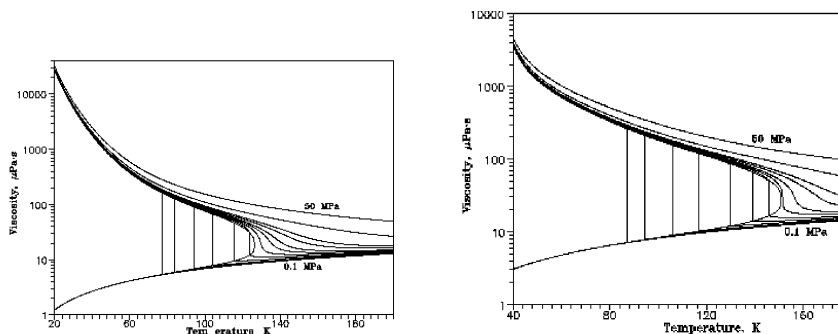


Fig. -1.6. Nitrogen (left) and Argon (right) viscosity as a function of the temperature and pressure after Lemmon and Jacobsen.

Figure 1.6 demonstrates that viscosity increases slightly with pressure, but this variation is negligible for most engineering problems. Well above the critical point, both materials are only a function of the temperature. On the liquid side below the critical point, the pressure has minor effect on the viscosity. It must be stress that the viscosity in the dome is meaningless. There is no such a thing of viscosity at 30% liquid. It simply depends on the structure of the flow as will be discussed in the chapter on multi phase flow. The lines in the above diagrams are only to show constant pressure lines. Oils have the greatest increase of viscosity with pressure which is a good thing for many engineering purposes.

1.5.2 Non-Newtonian Fluids

In equation (1.5), the relationship between the velocity and the shear stress was assumed to be linear. Not all the materials obey this relationship. There is a large class of materials which shows a non-linear relationship with velocity for any shear stress. This class of materials can be approximated by a single polynomial term that is $a = bx^n$. From the physical point of view, the coefficient depends on the velocity gradient. This relationship is referred to as power relationship

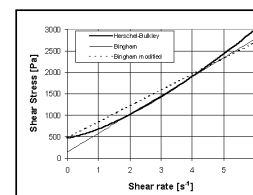


Fig. -1.7. The shear stress as a function of the shear rate.

and it can be written as

$$\tau = K \overbrace{\left(\frac{dU}{dx}\right)^{n-1}}^{\text{viscosity}} \left(\frac{dU}{dx}\right) \quad (1.13)$$

The new coefficients (n, K) in equation (1.13) are constant. When $n = 1$ equation represent Newtonian fluid and K becomes the familiar μ . The viscosity coefficient is always positive. When n , is above one, the liquid is dilatant. When n is below one, the fluid is pseudoplastic. The liquids which satisfy equation (1.13) are referred to as purely viscous fluids. Many fluids satisfy the above equation. Fluids that show increase in the viscosity (with increase of the shear) referred to as thixotropic and those that show decrease are called reoplectic fluids (see Figure 1.5).

Materials which behave up to a certain shear stress as a solid and above it as a liquid are referred as Bingham liquids. In the simple case, the “liquid side” is like Newtonian fluid for large shear stress. The general relationship for simple Bingham flow is

$$\tau_{xy} = -\mu \pm \tau_0 \quad \text{if } |\tau_{yx}| > \tau_0 \quad (1.14)$$

$$\frac{dU_x}{dy} = 0 \quad \text{if } |\tau_{yx}| < \tau_0 \quad (1.15)$$

There are materials that simple Bingham model does not provide dequate explanation and a more sophisticate model is required. The Newtonian part of the model has to be replaced by power liquid. For example, according to Ferraris at e^3 concrete behaves as shown in Figure 1.7. However, for most practical purposes, this kind of figures isn't used in regular engineering practice.

1.5.3 Kinematic Viscosity

The kinematic viscosity is another way to look at the viscosity. The reason for this new definition is that some experimental data are given in this form. These results also explained better using the new definition. The kinematic viscosity embraces both the viscosity and density properties of a fluid. The above equation shows that the dimensions of ν to be square meter per second, $[m^2/sec]$, which are acceleration units (a combination of kinematic terms). This fact explains the name “kinematic” viscosity. The kinematic viscosity is defined as

$$\nu = \frac{\mu}{\rho} \quad (1.16)$$

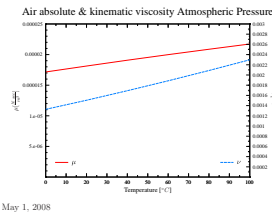


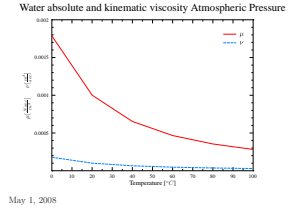
Fig. -1.8. Air viscosity as a function of the temperature.

³C. Ferraris, F. de Larrard and N. Martys, Materials Science of Concrete VI, S. Mindess and J. Skalny, eds., 215-241 (2001)

The gas density decreases with the temperature. However, The increase of the absolute viscosity with the temperature is enough to overcome the increase of density and thus, the kinematic viscosity also increase with the temperature for many materials.

1.5.4 Estimation of The Viscosity

The absolute viscosity of many fluids relatively doesn't change with the pressure but very sensitive to temperature. For isothermal flow, the viscosity can be considered constant in many cases. The variations of air and water as a function of the temperature at atmospheric pressure are plotted in Figures 1.8 and 1.9.



May 1, 2008

Some common materials (pure and mixture) have expressions that provide an estimate. For many gases, Sutherland's equation is used and according to the literature, provides reasonable results⁴ for the range of $-40^{\circ}C$ to $1600^{\circ}C$

$$\mu = \mu_0 \frac{0.555 T_{i0} + Suth}{0.555 T_{in} + Suth} \left(\frac{T}{T_0} \right)^{\frac{3}{2}} \quad (1.17)$$

Where

Example 1.3:

Calculate the viscosity of air at 800K based on Sutherland's equation. Use the data provide in Table 1.1.

SOLUTION

Applying the constants from Suthelnd's table provides

$$\mu = 0.00001827 \times \frac{0.555 \times 524.07 + 120}{0.555 \times 800 + 120} \times \left(\frac{800}{524.07} \right)^{\frac{3}{2}} \sim 2.51 \cdot 10^{-5} \left[\frac{N \cdot sec}{m^2} \right]$$

The viscosity increases almost by 40%. The observed viscosity is about $\sim 3.710^{-5} \left[\frac{N \cdot sec}{m^2} \right]$.

End Solution

⁴This author is ambivalent about this statement.

Material	Chemical formula	Sutherland	$T_{iO}[K]$	$\mu_0(N\ sec/m^2)$
ammonia	NH_3	370	527.67	0.00000982
standard air		120	524.07	0.00001827
carbon dioxide	CO_2	240	527.67	0.00001480
carbon monoxide	CO	118	518.67	0.00001720
hydrogen	H_2	72	528.93	0.0000876
nitrogen	N_2	111	540.99	0.0001781
oxygen	O_2	127	526.05	0.0002018
sulfur dioxide	SO_2	416	528.57	0.0001254

Table -1.1. The list for Sutherland's equation coefficients for selected materials.

Substance	Chemical formula	Temperature $T [^{\circ}C]$	Viscosity [$\frac{N\ sec}{m^2}$]
	$i - C_4 H_{10}$	23	0.0000076
	CH_4	20	0.0000109
	CO_2	20	0.0000146
oxygen	O_2	20	0.0000203
mercury vapor	Hg	380	0.0000654

Table -1.2. Viscosity of selected gases.

Table -1.3. Viscosity of selected liquids.

Chemical component	Chemical formula	Temperature T [$^{\circ}C$]	Viscosity [$\frac{N \cdot sec}{m^2}$]
	$(C_2H_5)_2O$	20	0.000245
	C_6H_6	20	0.000647
	Br_2	26	0.000946
	C_2H_5OH	20	0.001194
	Hg	25	0.001547
	H_2SO_4	25	0.01915
Olive Oil		25	0.084
Castor Oil		25	0.986
Clucose		25	5-20
Corn Oil		20	0.072
SAE 30		-	0.15-0.200
SAE 50		$\sim 25^{\circ}C$	0.54
SAE 70		$\sim 25^{\circ}C$	1.6
Ketchup		$\sim 20^{\circ}C$	0,05
Ketchup		$\sim 25^{\circ}C$	0,098
Benzene		$\sim 20^{\circ}C$	0.000652
Firm glass		-	$\sim 1 \times 10^7$
Glycerol		20	1.069

Liquid Metals

Liquid metal can be considered as a Newtonian fluid for many applications. Furthermore, many aluminum alloys are behaving as a Newtonian liquid until the first solidification appears (assuming steady state thermodynamics properties). Even when there is a solidification (mushy zone), the metal behavior can be estimated as a Newtonian material (further reading can be done in this author's book "Fundamentals of Die Casting Design"). Figure 1.10 exhibits several liquid metals (from The Reactor Handbook, Vol. Atomic Energy Commission AECD-3646 U.S. Government Printing Office, Washington D.C. May 1995 p. 258.)

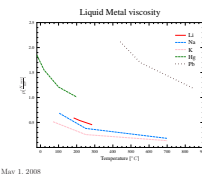


Fig. -1.10. Liquid metals viscosity as a function of the temperature.

The General Viscosity Graphs

Chemical component	Molecular Weight	T_c [K]	P_c [Bar]	μ_c [$\frac{N \text{ sec}}{m^2}$]
H_2	2.016	33.3	12.9696	3.47
He	4.003	5.26	2.289945	2.54
Ne	20.183	44.5	27.256425	15.6
Ar	39.944	151	48.636	26.4
Xe	131.3	289.8	58.7685	49.
Air "mixed"	28.97	132	36.8823	19.3
CO_2	44.01	304.2	73.865925	19.0
O_2	32.00	154.4	50.358525	18.0
C_2H_6	30.07	305.4	48.83865	21.0
CH_4	16.04	190.7	46.40685	15.9
Water		647.096 K	22.064 [MPa]	

Table -1.4. The properties at the critical stage and their values of selected materials.

In case "ordinary" fluids where information is limit, Hougen et al suggested to use graph similar to compressibility chart. In this graph, if one point is well documented, other points can be estimated. Furthermore, this graph also shows the trends. In Figure 1.11 the relative viscosity $\mu_r = \mu/\mu_c$ is plotted as a function of relative temperature, T_r . μ_c is the viscosity at critical condition and μ is the viscosity at any given condition. The lines of constant relative pressure, $P_r = P/P_c$ are drawn. The lower pressure is, for practical purpose, ~ 1 [bar].

The critical pressure can be evaluated in the following three ways. The simplest way is by obtaining the data from Table 1.4 or similar information. The second way, if the information is available and is close enough to the critical point, then the critical viscosity is obtained as

$$\mu_c = \frac{\overbrace{\mu}^{\text{given}}}{\underbrace{\mu_r}_{\text{figure 1.11}}} \quad (1.18)$$

The third way, when none is available, is by utilizing the following approximation

$$\mu_c = \sqrt{M T_c \tilde{v}_c^{2/3}} \quad (1.19)$$

Where \tilde{v}_c is the critical molecular volume and M is molecular weight. Or

$$\mu_c = \sqrt{M P_c^{2/3} T_c^{-1/6}} \quad (1.20)$$

Calculate the reduced pressure and the reduced temperature and from the Figure 1.11 obtain the reduced viscosity.

Example 1.4:

Estimate the viscosity of oxygen, O_2 at $100^\circ C$ and 20 [Bar].

SOLUTION

The critical condition of oxygen are $P_c = 50.35[\text{Bar}]$ $T_c = 154.4$ $\mu_c = 18 \left[\frac{\text{N}\cdot\text{sec}}{\text{m}^2} \right]$ The value of the reduced temperature is

$$T_r \sim \frac{373.15}{154.4} \sim 2.41$$

The value of the reduced pressure is

$$P_r \sim \frac{20}{50.35} \sim 0.4$$

From Figure 1.11 it can be obtained $\mu_r \sim 1.2$ and the predicted viscosity is

$$\mu = \mu_c \overbrace{\left(\frac{\mu}{\mu_c} \right)}^{\text{Table}} = 18 \times 1.2 = 21.6[\text{N}\cdot\text{sec}/\text{m}^2]$$

The observed value is $24[\text{N}\cdot\text{sec}/\text{m}^2]$ ⁵.

End Solution

Viscosity of Mixtures

In general the viscosity of liquid mixture has to be evaluated experimentally. Even for homogeneous mixture, there isn't silver bullet to estimate the viscosity. In this book, only the mixture of low density gases is discussed for analytical expression. For most cases, the following Wilke's correlation for gas at low density provides a result in a reasonable range.

$$\mu_{mix} = \sum_{i=1}^n \frac{x_i \mu_i}{\sum_{j=1}^n x_j \Phi_{ij}} \quad (1.21)$$

where Φ_{ij} is defined as

$$\Phi_{ij} = \frac{1}{\sqrt{8}} \sqrt{1 + \frac{M_i}{M_j}} \left(1 + \sqrt{\frac{\mu_i}{\mu_j}} \sqrt{\frac{M_j}{M_i}} \right)^2 \quad (1.22)$$

Here, n is the number of the chemical components in the mixture. x_i is the mole fraction of component i , and μ_i is the viscosity of component i . The subscript i should be used for the j index. The dimensionless parameter Φ_{ij} is equal to one when $i = j$. The mixture viscosity is highly nonlinear function of the fractions of the components.

Example 1.5:

Calculate the viscosity of a mixture (air) made of 20% oxygen, O_2 and 80% nitrogen N_2 for the temperature of 20°C .

⁵Kyama, Makita, Rev. Physical Chemistry Japan Vol. 26 No. 2 1956.

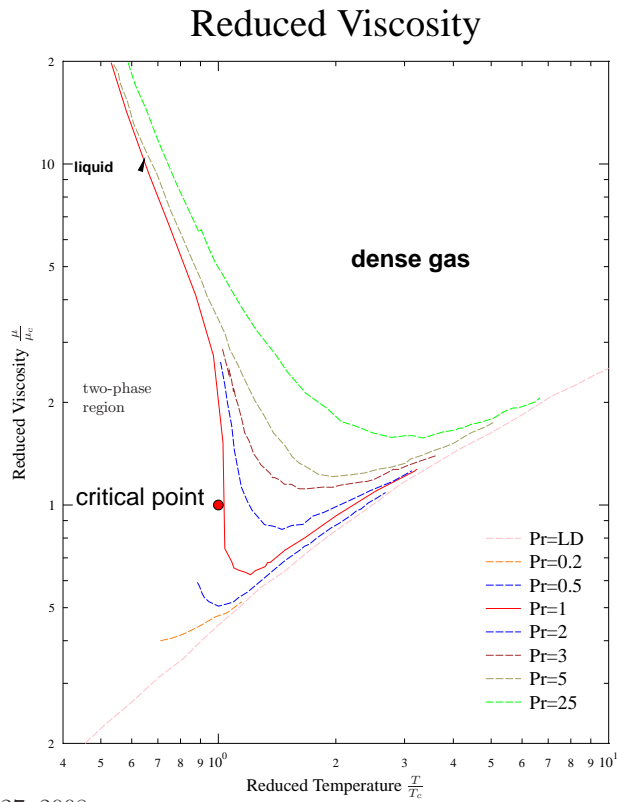
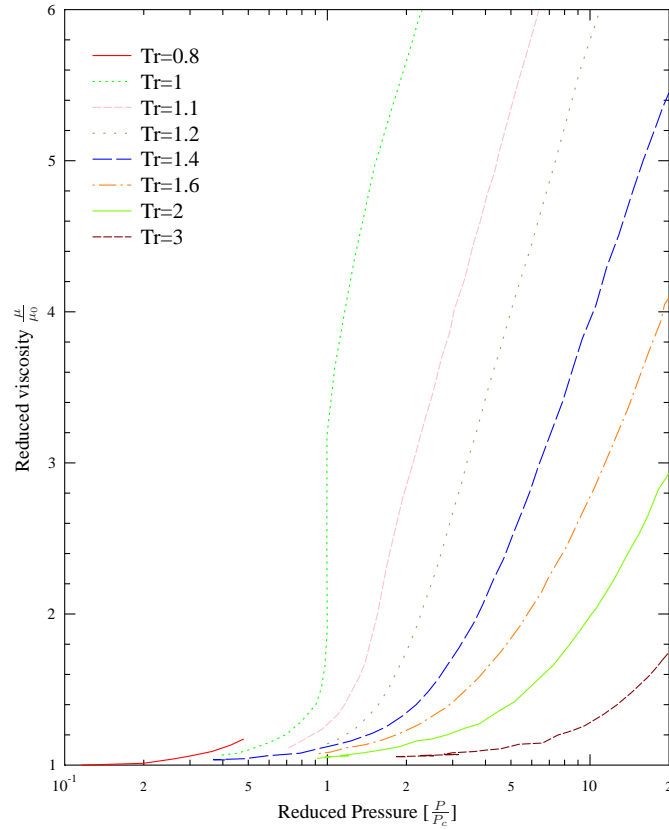


Fig. -1.11. Reduced viscosity as function of the reduced temperature.

SOLUTION

The following table summarize the known details

i	Component	Molecular Weight, M	Mole Fraction, x	Viscosity, μ
1	O_2	32.	0.2	0.0000203
2	N_2	28.	0.8	0.00001754



June 2, 2008

Fig. -1.12. Reduced viscosity as function of the reduced temperature.

i	j	M_i/M_j	μ_i/μ_j	Φ_{ij}
1	1	1.0	1.0	1.0
	2	1.143	1.157	1.0024
2	1	0.875	.86	0.996
	2	1.0	1.0	1.0

$$\mu_{mix} \sim \frac{0.2 \times 0.0000203}{0.2 \times 1.0 + 0.8 \times 1.0024} + \frac{0.8 \times 0.00001754}{0.2 \times 0.996 + 0.8 \times 1.0} \sim 0.0000181 \left[\frac{N \cdot sec}{m^2} \right]$$

The observed value is $\sim 0.0000182 \left[\frac{N \cdot sec}{m^2} \right]$.

End Solution

In very low pressure, in theory, the viscosity is only a function of the temperature with a “simple” molecular structure. For gases with very long molecular structure or complexity structure these formulas cannot be applied. For some mixtures of two liquids it was observed that at a low shear stress, the viscosity is dominated by a liquid with high viscosity and at high shear stress to be dominated by a liquid with the low viscosity liquid. The higher viscosity is more dominate at low shear stress. Reiner and Phillippoff suggested the following formula

$$\frac{dU_x}{dy} = \left(\frac{1}{\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \left(\frac{\tau_{xy}}{\tau_s}\right)^2}} \right) \tau_{xy} \quad (1.23)$$

Where the term μ_∞ is the experimental value at high shear stress. The term μ_0 is the experimental viscosity at shear stress approaching zero. The term τ_s is the characteristic shear stress of the mixture. An example for values for this formula, for Molten Sulfur at temperature $120^\circ C$ are $\mu_\infty = 0.0215 \left(\frac{N \text{ sec}}{m^2}\right)$, $\mu_0 = 0.00105 \left(\frac{N \text{ sec}}{m^2}\right)$, and $\tau_s = 0.0000073 \left(\frac{kN}{m^2}\right)$. This equation (1.23) provides reasonable value only up to $\tau = 0.001 \left(\frac{kN}{m^2}\right)$.

Figure 1.12 can be used for a crude estimate of dense gases mixture. To estimate the viscosity of the mixture with n component Hougen and Watson's method for pseudocritical properties is adapted. In this method the following are defined as mixed critical pressure as

$$P_{c_{mix}} = \sum_{i=1}^n x_i P_{c_i} \quad (1.24)$$

the mixed critical temperature is

$$T_{c_{mix}} = \sum_{i=1}^n x_i T_{c_i} \quad (1.25)$$

and the mixed critical viscosity is

$$\mu_{c_{mix}} = \sum_{i=1}^n x_i \mu_{c_i} \quad (1.26)$$

Example 1.6:

An inside cylinder with a radius of 0.1 [m] rotates concentrically within a fixed cylinder of 0.101 [m] radius and the cylinders length is 0.2 [m]. It is given that a moment of 1 [N × m] is required to maintain an angular velocity of 31.4 revolution per second (these number represent only academic question not real value of actual liquid). Estimate the liquid viscosity used between the cylinders.

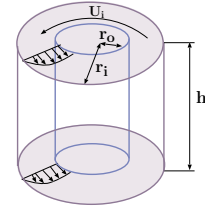


Fig. -1.13. Concentrating cylinders with the rotating inner cylinder.

SOLUTION

The moment or the torque is transmitted through the liquid to the outer cylinder. Control volume around the inner cylinder shows that moment is a function of the area and shear stress. The shear stress calculations can be estimated as a linear between the two concentric cylinders. The velocity at the inner cylinders surface is

$$U_i = r \omega = 0.1 \times 31.4 [\text{rad/second}] = 3.14 [\text{m/s}] \quad (1.VI.a)$$

The velocity at the outer cylinder surface is zero. The velocity gradient may be assumed to be linear, hence,

$$\frac{dU}{dr} \cong \frac{0.1 - 0}{0.101 - 0.1} = 100 \text{sec}^{-1} \quad (1.VI.b)$$

The used moment is

$$M = \underbrace{2\pi r_i h}_A \underbrace{\mu \frac{dU}{dr}}_{\tau} \underbrace{r_i}_\ell \quad (1.VI.c)$$

or the viscosity is

$$\mu = \frac{M}{2\pi r_i^2 h \frac{dU}{dr}} = \frac{1}{2 \times \pi \times 0.1^2 \times 0.2 \times 100} = \quad (1.VI.d)$$

End Solution

Example 1.7:

A square block weighing 1.0 [kN] with a side surfaces area of 0.1 [m²] slides down an incline surface with an angle of 20° C. The surface is covered with oil film. The oil creates a distance between the block and the inclined surface of 1 × 10⁻⁶ [m]. What is the speed of the block at steady state? Assuming a linear velocity profile in the oil and that the whole oil is under steady state. The viscosity of the oil is 3 × 10⁻⁵ [m²/sec].

SOLUTION

The shear stress at the surface is estimated for steady state by

$$\tau = \mu \frac{dU}{dx} = 3 \times 10^{-5} \times \frac{U}{1 \times 10^{-6}} = 30 U \quad (1.VII.a)$$

The total friction force is then

$$f = \tau A = 0.1 \times 30U = 3U \quad (1.VII.b)$$

The gravity force that acting against the friction is equal to the friction hence

$$F_g = f = 3U \implies U = \frac{mg \sin 20^\circ}{3} \quad (1.VII.c)$$

Or the solution is

$$U = \frac{1 \times 9.8 \times \sin 20^\circ}{3} \quad (1.VII.d)$$

End Solution

Example 1.8:

Develop an expression to estimate of the torque required to rotate a disc in a narrow gap. The edge effects can be neglected. The gap is given and equal to δ and the rotation speed is ω . The shear stress can be assumed to be linear.

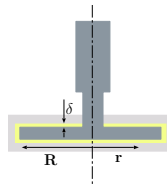


Fig. -1.14. Rotating disc in a steady state.

SOLUTION

In this cases the shear stress is a function of the radius, r and an expression has to be developed. Additionally, the differential area also increases and is a function of r . The shear stress can be estimated as

$$\tau \cong \mu \frac{U}{\delta} = \mu \frac{\omega r}{\delta} \quad (1.VIII.a)$$

This torque can be integrated for the entire area as

$$T = \int_0^R r \tau dA = \int_0^R \underbrace{r}_{\ell} \underbrace{\mu \frac{\omega r}{\delta}}_{\tau} \underbrace{2\pi r dr}_{dA} \quad (1.VIII.b)$$

The results of the integration is

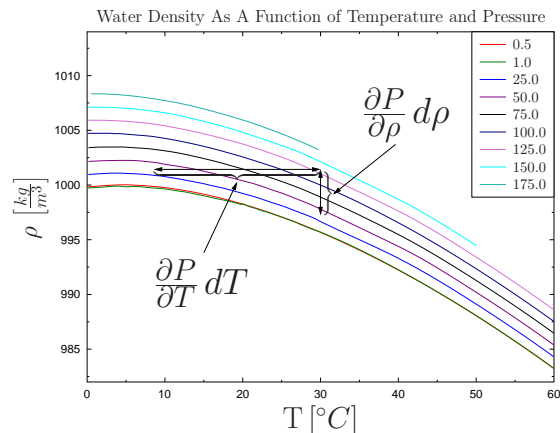
$$T = \frac{\pi \mu \omega R^4}{2\delta} \quad (1.VIII.c)$$

End Solution

1.6 Fluid Properties

The fluids have many properties which are similar to solid. A discussion of viscosity and surface tension should be part of this section but because special importance these topics have separate sections. The rest of the properties lumped into this section.

1.6.1 Fluid Density



March 15, 2011

Fig. -1.15. Water density as a function of temperature for various pressure. This figure illustrates the typical situations like the one that appear in Example 1.9

The density is a property that is simple to analyzed and understand. The density is related to the other state properties such temperature and pressure through the equation of state or similar. Examples to describe the usage of property are provided.

Example 1.9:

A steel tank filled with water undergoes heating from 10°C to 50°C . The initial pressure can be assumed to atmospheric. Due to the change temperature the tank, (strong steel structure) undergoes linear expansion of 8×10^{-6} per $^{\circ}\text{C}$. Calculate the pressure at the end of the process. E denotes the Young's modulus⁶. Assume that the Young modulus of the water is $2.15 \times 10^9 (\text{N}/\text{m}^2)$ ⁷. State your assumptions.

SOLUTION

⁶The definition of Young's modulus is $E = \frac{\sigma}{\epsilon}$ where in this case σ can be estimated as the pressure change. The definition of ϵ is the ratio length change to total length $\Delta L/L$.

⁷This value is actually of Bulk modulus.

The expansion of the steel tank will be due to two contributions: one due to the thermal expansion and one due to the pressure increase in the tank. For this example, it is assumed that the expansion due to pressure change is negligible. The tank volume change under the assumptions state here but in the same time the tank walls remain straight. The new density is

$$\rho_2 = \frac{\rho_1}{\underbrace{(1 + \alpha \Delta T)^3}_{\text{thermal expansion}}} \quad (1.IX.a)$$

The more accurate calculations require looking into the steam tables. As estimated value of the density using Young's modulus and $V_2 \propto (L_2)^3$ ⁸.

$$\rho_2 \propto \frac{1}{(L_2)^3} \implies \rho_2 \cong \frac{m}{\left(L_1 \left(1 - \frac{\Delta P}{E}\right)\right)^3} \quad (1.IX.b)$$

It can be noticed that $\rho_1 \cong m/L_1^3$ and thus

$$\frac{\rho_1}{(1 + \alpha \Delta T)^3} = \frac{\rho_1}{\left(1 - \frac{\Delta P}{E}\right)^3} \quad (1.IX.c)$$

The change is then

$$1 + \alpha \Delta T = 1 - \frac{\Delta P}{E} \quad (1.IX.d)$$

Thus the final pressure is

$$P_2 = P_1 - E \alpha \Delta T \quad (1.IX.e)$$

In this case, what happen when the value of $P_1 - E \alpha \Delta T$ becomes negative or very very small? The basic assumption falls and the water evaporates.

If the expansion of the water is taken into account then the change (increase) of water volume has to be taken into account. The tank volume was calculated earlier and since the claim of "strong" steel the volume of the tank is only effected by the temperature.

$$\left. \frac{V_2}{V_1} \right|_{\text{tank}} = (1 + \alpha \Delta T)^3 \quad (1.IX.f)$$

The volume of the water undergoes also a change and is a function of the temperature and pressure. The water pressure at the end of the process is unknown but the volume is known. Thus, the density at end is also known

$$\rho_2 = \frac{m_w}{T_2|_{\text{tank}}} \quad (1.IX.g)$$

The pressure is a function volume and the temperature $P = P(v, T)$ thus

$$dP = \overbrace{\left(\frac{\partial P}{\partial v}\right)}^{\sim \beta_v} dv + \overbrace{\left(\frac{\partial P}{\partial T}\right)}^{\sim E} dT \quad (1.IX.h)$$

⁸This leads $E(L_2 - L_1) = \Delta P L_1$. Thus, $L_2 = L_1(1 - \Delta P/E)$

As approximation it can written as

$$\Delta P = \beta_v \Delta v + E \Delta T \quad (1.IX.i)$$

Substituting the values results for

$$\Delta P = \frac{0.0002}{\Delta \rho} + 2.15 \times 10^9 \Delta T \quad (1.IX.j)$$

Notice that density change, $\Delta \rho < 0$.

End Solution

1.6.2 Bulk Modulus

Similar to solids (hook's law), liquids have a property that describes the volume change as results of pressure change for constant temperature. It can be noted that this property is not the result of the equation of state but related to it. Bulk modulus is usually obtained from experimental or theoretical or semi theoretical (theory with experimental work) to fit energy–volume data. Most (theoretical) studies are obtained by uniformly changing the unit cells in global energy variations especially for isotropic systems (where the molecules has a structure with cubic symmetries). The bulk modulus is a measure of the energy can be stored in the liquid. This coefficient is analogous to the coefficient of spring. The reason that liquid has different coefficient is because it is three dimensional verse one dimension that appear in regular spring.

The bulk modulus is defined as

$$B_T = -v \left(\frac{\partial P}{\partial v} \right)_T \quad (1.27)$$

Using the identity of $v = 1/\rho$ transfers equation (1.27) into

$$B_T = \rho \left(\frac{\partial P}{\partial \rho} \right)_T \quad (1.28)$$

The bulk modulus for several selected liquids is presented in Table 1.5.

Table -1.5. The bulk modulus for selected material with the critical temperature and pressure *na* → not available and *n.f* → not found (exist but was not found in the literature).

Chemical component	Bulk Modulus $10^9 \frac{N}{m}$	T_c	P_c
Acetic Acid	2.49	593K	57.8 [Bar]
Acetone	0.80	508 K	48 [Bar]
Benzene	1.10	562 K	4.74 [MPa]
Carbon Tetrachloride	1.32	556.4 K	4.49 [MPa]

Table -1.5. Bulk modulus for selected materials (continue)

Chemical component	Bulk Modulus $10^9 \frac{N}{m}$	T_c	P_c
Ethyl Alcohol	1.06	514 K	6.3 [Mpa]
Gasoline	1.3	nf	nf
Glycerol	4.03-4.52	850 K	7.5 [Bar]
Mercury	26.2-28.5	1750 K	172.00 [MPa]
Methyl Alcohol	0.97	Est 513	Est 78.5 [Bar]
Nitrobenzene	2.20	nf	nf
Olive Oil	1.60	nf	nf
Paraffin Oil	1.62	nf	nf
SAE 30 Oil	1.5	na	na
Seawater	2.34	na	na
Toluene	1.09	591.79 K	4.109 [MPa]
Turpentine	1.28	na	na
Water	2.15-2.174	647.096 K	22.064 [MPa]

In the literature, additional expansions for similar parameters are defined. The thermal expansion is defined as

$$\beta_P = \frac{1}{v} \left(\frac{\partial v}{\partial T} \right)_P \quad (1.29)$$

This parameter indicates the change of volume due to temperature change when the pressure is constant. Another definition is referred as coefficient of tension and it is defined as

$$\beta_v = \frac{1}{P} \left(\frac{\partial P}{\partial T} \right)_v \quad (1.30)$$

This parameter indicates the change of the pressure due to the change of temperature (where $v = \text{constant}$). These definitions are related to each other. This relationship is obtained by the observation that the pressure as a function of the temperature and specific volume as

$$P = f(T, v) \quad (1.31)$$

The full pressure derivative is

$$dP = \left(\frac{\partial P}{\partial T} \right)_v dT + \left(\frac{\partial P}{\partial v} \right)_T dv \quad (1.32)$$

On constant pressure lines, $dP = 0$, and therefore equation (1.32) reduces

$$0 = \left(\frac{\partial P}{\partial T} \right)_v dT + \left(\frac{\partial P}{\partial v} \right)_T dv \quad (1.33)$$

From equation (1.33) follows that

$$\left. \frac{dv}{dT} \right|_{P=\text{const}} = - \frac{\left(\frac{\partial P}{\partial T} \right)_v}{\left(\frac{\partial P}{\partial v} \right)_T} \quad (1.34)$$

Equation (1.34) indicates that relationship for these three coefficients is

$$\beta_T = - \frac{\beta_v}{\beta_P} \quad (1.35)$$

The last equation (1.35) sometimes is used in measurement of the bulk modulus.

The increase of the pressure increases the bulk modulus due to the molecules increase of the rejecting forces between each other when they are closer. In contrast, the temperature increase results in reduction of the bulk of modulus because the molecular are further away.

Example 1.10:

Calculate the modulus of liquid elasticity that reduced 0.035 per cent of its volume by applying a pressure of 5[Bar] in a slow process.

SOLUTION

Using the definition for the bulk modulus

$$\beta_T = -v \frac{\partial P}{\partial v} \simeq \frac{v}{\Delta v} \Delta P = \frac{5}{0.00035} \simeq 14285.714[\text{Bar}]$$

End Solution

Example 1.11:

Calculate the pressure needed to apply on water to reduce its volume by 1 per cent. Assume the temperature to be 20°C.

SOLUTION

Using the definition for the bulk modulus

$$\Delta P \sim \beta_T \frac{\Delta v}{v} \sim 2.15 \cdot 10^9 \cdot 0.01 = 2.15 \cdot 10^7 [\text{N/m}^2] = 215[\text{Bar}]$$

End Solution

Example 1.12:

1.6. FLUID PROPERTIES

Two layers of two different liquids are contained in a very solid tank. Initially the pressure in the tank is P_0 . The liquids are compressed due to the pressure increases. The new pressure is P_1 . The area of the tank is A and liquid A height is h_1 and liquid B height is h_2 . Estimate the change of the heights of the liquids depicted in the Figure 1.16. State your assumptions.

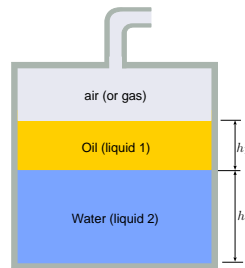


Fig. -1.16. Two liquid layers under pressure.

SOLUTION

The volume change in a liquid is

$$B_T \cong \frac{\Delta P}{\Delta V/V} \tag{1.XII.a}$$

Hence the change for the any liquid is

$$\Delta h = \frac{\Delta P}{A B_T/V} = \frac{h \Delta P}{B_T} \tag{1.XII.b}$$

The total change when the hydrostatic pressure is ignored.

$$\Delta h_{1+2} = \Delta P \left(\frac{h_1}{B_{T1}} + \frac{h_2}{B_{T2}} \right) \tag{1.XII.c}$$

End Solution

Example 1.13:

In the Internet the following problem (here with \LaTeX modification) was posted which related to Pushka equation.

A cylindrical steel pressure vessel with volume 1.31 m^3 is to be tested. The vessel is entirely filled with water, then a piston at one end of the cylinder is pushed in until the pressure inside the vessel has increased by 1000 kPa. Suddenly, a safety plug on the top bursts. How many liters of water come out?

Relevant equations and data suggested by the user were: $B_T = 0.2 \times 10^{10} \text{ N/m}^2$, $P_1 = P_0 + \rho g h$, $P_1 = -B_T \Delta V/V$ with the suggested solution of

"I am assuming that I have to look for ΔV as that would be the water that comes out causing the change in volume."

$$\Delta V = \frac{-V \Delta P}{B_T} = -1.31(1000)/(0.2 \times 10^{10}) \Delta V = 6.55 * 10^{-7}$$

Another user suggest that:

We are supposed to use the bulk modulus from our textbook, and that one is 0.2×10^{10} .

Anything else would give a wrong answer in the system. So with this bulk modulus, is 0.655L right?

In this post several assumptions were made. What is a better way to solve this problem.

SOLUTION

It is assumed that this process can be between two extremes: one isothermal and one isentropic. The assumption of isentropic process is applicable after a shock wave that travel in the tank. If the shock wave is ignored (too advance material for this book⁹), the process is isentropic. The process involve some thermodynamics identities to be connected. Since the pressure is related or a function of density and temperature it follows that

$$P = P(\rho, T) \quad (1.XIII.a)$$

Hence the full differential is

$$dP = \left. \frac{\partial P}{\partial \rho} \right|_T d\rho + \left. \frac{\partial P}{\partial T} \right|_\rho dT \quad (1.XIII.b)$$

Equation (1.XIII.b) can be multiplied by ρ/P to be

$$\frac{\rho dP}{P} = \frac{1}{P} \left(\overbrace{\rho \left. \frac{\partial P}{\partial \rho} \right|_T}^{B_T} d\rho \right) + \rho \left(\overbrace{\left. \frac{1}{P} \frac{\partial P}{\partial T} \right|_\rho}^{\beta_v} dT \right) \quad (1.XIII.c)$$

The definitions that were provided before can be used to write

$$\frac{\rho dP}{P} = \frac{1}{P} B_T d\rho + \rho \beta_v dT \quad (1.XIII.d)$$

The infinitesimal change of density will be then

$$\frac{1}{P} B_T d\rho = \frac{\rho dP}{P} - \rho \beta_v dT \quad (1.XIII.e)$$

or

$$d\rho = \frac{\rho dP}{B_T} - \frac{\rho P \beta_v dT}{B_T} \quad (1.XIII.f)$$

Thus, the calculation that were provide on line need to have corrections by subtracting the second terms.

End Solution

⁹The shock wave velocity is related to square of elasticity of the water. Thus the characteristic time for the shock is S/c when S is a typical dimension of the tank and c is speed of sound of the water in the tank.

Example 1.14:

The hydrostatic pressure was neglected in example 1.12. In some places the ocean depth is many kilometers (the deepest places is more than 10 kilometers). For this example, calculate the density change in the bottom of 10 kilometers using two methods. In one method assume that the density is remain constant until the bottom. In the second method assume that the density is a function of the pressure.

SOLUTION

For the the first method the density is

$$B_T \cong \frac{\Delta P}{\Delta V/V} \implies \Delta V = V \frac{\Delta P}{B_T} \quad (1.XIV.a)$$

The density at the surface is $\rho = m/V$ and the density at point x from the surface the density is

$$\rho(x) = \frac{m}{V - \Delta V} \implies \rho(x) = \frac{m}{V - V \frac{\Delta P}{B_T}} \quad (1.XIV.b)$$

In the Chapter on static it will be shown that the change pressure is

$$\Delta P = g \int_0^x \rho(x) dx \quad (1.XIV.c)$$

Combining equation (1.XIV.b) with equation (1.XIV.c) yields

$$\rho(x) = \frac{m}{V - V \frac{g \int_0^x \rho(x) dx}{B_T}} \quad (1.XIV.d)$$

Equation can be rearranged to be

$$\rho(x) = \frac{m}{V \left(1 - \frac{g}{B_T} \int_0^x \rho(x) dx \right)} \implies \rho(x) = \frac{\rho_0}{\left(1 - \frac{g}{B_T} \int_0^x \rho(x) dx \right)} \quad (1.XIV.e)$$

Equation (1.XIV.e) is an integral equation which is discussed in the appendix¹⁰. . It is convenient to change further equation (1.XIV.e) to

$$1 - \frac{g}{B_T} \int_0^x \rho(x) dx = \frac{\rho_0}{\rho(x)} \quad (1.XIV.f)$$

The integral equation (1.XIV.f) can be converted to differential equation when the two sides under differentiation

$$\frac{g}{B_T} \rho(x) + \frac{\rho_0}{\rho(x)^2} \frac{d\rho(x)}{dx} = 0 \quad (1.XIV.g)$$

¹⁰Under construction

or

$$\frac{g \rho(x)^3}{B_T \rho_0} + \frac{d\rho(x)}{dx} = 0 \quad (1.XIV.h)$$

The solution is

$$\frac{\rho_0 B_T}{2 g \rho^2} = x + c \quad (1.XIV.i)$$

or rearranged as

$$\rho = \sqrt{\frac{\rho_0 B_T}{2 g (x + c)}} \quad (1.XIV.j)$$

The integration constant can be found by the fact that the density at the $x = 0$ is ρ_0

$$\rho_0 = \sqrt{\frac{\rho_0 B_T}{2 g (c)}} \implies c = \frac{B_T}{2 g \rho_0} \quad (1.XIV.k)$$

Substituting the integration constant, the solution is

$$\frac{\rho}{\rho_0} = \sqrt{\frac{\rho_0 B_T}{2 g \rho_0 x + B_T}} \quad (1.XIV.l)$$

In the “constant” density approach, the density at the bottom using equation (1.XIV.e) was

$$\rho = \frac{\rho_0}{1 - \frac{g}{B_T} g \rho_0 x} \implies \frac{\rho_0 B_T}{B_T - g \rho_0 x} \quad (1.XIV.m)$$

End Solution

Advance material can be skipped

Example 1.15:

Water in deep sea undergoes compression due to hydrostatic pressure. That is the density is function of the depth. For constant bulk modulus, it was shown in “Fundamentals of Compressible Flow” by this author that the speed of sound is

$$c = \sqrt{\frac{B_T}{\rho}} \quad (1.XV.a)$$

Calculate the time it take for a sound wave to propagate perpendicularly to the surface to a depth D (perpendicular to the straight surface). Assume that no variation of the temperature. For the purpose of this exercise, the salinity can be completely ignored.

SOLUTION

The equation for the sound speed is taken here as correct for very local point. However,

the density is different for every point since the density varies and the density is a function of the depth. The speed of sound at any depth point, x , is

$$c = \sqrt{\frac{B_T}{\rho_0 B_T}} = \sqrt{\frac{B_T - g \rho_0 x}{\rho_0}} \quad (1.XV.b)$$

The time the sound travel a small interval distance, dx is

$$d\tau = \frac{dx}{\sqrt{\frac{B_T - g \rho_0 x}{\rho_0}}} \quad (1.XV.c)$$

The time takes for the sound the travel the whole distance is the integration of infinitesimal time

$$t = \int_0^D \frac{dx}{\sqrt{\frac{B_T - g \rho_0 x}{\rho_0}}} \quad (1.XV.d)$$

The solution of equation (1.XV.d) is

$$t = \sqrt{\rho_0} \left(2\sqrt{B_T} - 2\sqrt{B_T - D} \right) \quad (1.XV.e)$$

The time to travel according to the standard procedure is

$$t = \frac{D}{\sqrt{\frac{B_T}{\rho_0}}} = \frac{D\sqrt{\rho_0}}{\sqrt{B_T}} \quad (1.XV.f)$$

The ratio between the corrected estimated to the standard calculation is

$$\text{correction ratio} = \frac{\sqrt{\rho_0} (2\sqrt{B_T} - 2\sqrt{B_T - D})}{\frac{D\sqrt{\rho_0}}{\sqrt{B_T}}} \quad (1.XV.g)$$

End Solution

1.6.2.1 Bulk Modulus of Mixtures

In the discussion above it was assumed that the liquid is pure. In this short section a discussion about the bulk modulus averaged is presented. When more than one liquid are exposed to pressure the value of these two (or more liquids) can have to be added in special way. The definition of the bulk modulus is given by equation (1.27) or (1.28) and can be written (where the partial derivative can looks as delta Δ as

$$\partial V = \frac{V \partial P}{B_T} \cong \frac{V \Delta P}{B_T} \quad (1.36)$$

The total change is compromised by the change of individual liquids or phases if two materials are present. Even in some cases of emulsion (a suspension of small globules of one liquid in a second liquid with which the first will not mix) the total change is the summation of the individuals change. In case the total change isn't, in special mixture, another approach with taking into account the energy-volume is needed. Thus, the total change is

$$\partial V = \partial V_1 + \partial V_2 + \dots + \partial V_i \cong \Delta V_1 + \Delta V_2 + \dots + \Delta V_i \quad (1.37)$$

Substituting equation (1.36) into equation (1.37) results in

$$\partial V = \frac{V_1 \partial P}{B_{T1}} + \frac{V_2 \partial P}{B_{T2}} + \dots + \frac{V_i \partial P}{B_{Ti}} \cong \frac{V_1 \Delta P}{B_{T1}} + \frac{V_2 \Delta P}{B_{T2}} + \dots + \frac{V_i \Delta P}{B_{Ti}} \quad (1.38)$$

Under the main assumption in this model the total volume is comprised of the individual volume hence,

$$V = x_1 V + x_2 V + \dots + x_i V \quad (1.39)$$

Where x_1 , x_2 and x_i are the fraction volume such as $x_i = V_i/V$. Hence, using this identity and the fact that the pressure is change for all the phase uniformly equation (1.39) can be written as

$$\partial V = V \partial P \left(\frac{x_1}{B_{T1}} + \frac{x_2}{B_{T2}} + \dots + \frac{x_i}{B_{Ti}} \right) \cong V \Delta P \left(\frac{x_1}{B_{T1}} + \frac{x_2}{B_{T2}} + \dots + \frac{x_i}{B_{Ti}} \right) \quad (1.40)$$

Rearranging equation (1.40) yields

$$v \frac{\partial P}{\partial v} \cong v \frac{\Delta P}{\Delta v} = \frac{1}{\left(\frac{x_1}{B_{T1}} + \frac{x_2}{B_{T2}} + \dots + \frac{x_i}{B_{Ti}} \right)} \quad (1.41)$$

Equation (1.41) suggested an averaged new bulk modulus

$$B_{Tmix} = \frac{1}{\left(\frac{x_1}{B_{T1}} + \frac{x_2}{B_{T2}} + \dots + \frac{x_i}{B_{Ti}} \right)} \quad (1.42)$$

In that case the equation for mixture can be written as

$$v \frac{\partial P}{\partial v} = B_{Tmix} \quad (1.43)$$

¹¹To be added in the future the effect of change of chemical composition on bulk modulus.

1.6.2.2 When the Bulk Modulus is Important? and Hydraulics System

There are only several situations in which the bulk modulus is important. These situations include hydraulic systems, deep ocean (on several occasions), geology system like the Earth, Cosmology. The Pushka equation normally can address the situations in deep ocean and geological system. This author is not aware of any special issues that involve in Cosmology as opposed to geological system. The only issue that was not address is the effect on hydraulic systems. The hydraulic system normally refers to system in which a liquid is used to transmit forces (pressure) for surface of moving object (normally piston) to another. For theoretical or hypothetical liquids which moving one object (surface) results in movement of the other object under the condition that liquid volume is fix. when the liquid volume or density is function of the pressure (and temperature due the friction) the movement of the other object is unpredictable. For very accurate and rapid systems, the temperature and pressure varies during the operation. In practical situations, the commercial hydraulic fluid can change due to friction by 50°C . The bulk modulus for the same liquid change by more 60%. The change of the bulk modulus by this amount can change the response time significantly.

1.7 Surface Tension

The surface tension manifested itself by a rise or depression of the liquid at the free surface edge. Surface tension is also responsible for the creation of the drops and bubbles. It also responsible for the breakage of a liquid jet into other medium/phase to many drops (atomization). The surface tension is force per length and is measured by $[\text{N}/\text{m}]$ and is acting to stretch the surface.

Surface tension results from a sharp change in the density between two adjoined phases or materials. There is a common misconception for the source of the surface tension. In many (physics, surface tension, and fluid mechanics) books explained that the surface tension is a result from unbalance molecular cohesive forces. This explanation is wrong since it is in conflict with Newton's second law (see example ?). This erroneous explanation can be traced to Adam's book but earlier source may be found.

The relationship between the surface tension and the pressure on the two sides of the surface is based on geometry. Consider a small element of surface. The pressure on one side is P_i and the pressure on the other side is P_o . When the surface tension is constant, the horizontal forces cancel each other because symmetry. In the vertical

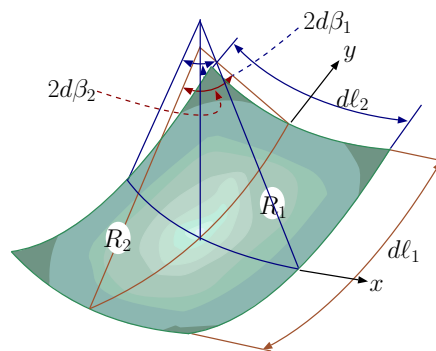


Fig. -1.17. Surface tension control volume analysis describing principles radii.

direction, the surface tension forces are pulling the surface upward. Thus, the pressure difference has to balance the surface tension. The forces in the vertical direction reads

$$(P_i - P_o) dl_1 dl_2 = \Delta P dl_1 dl_2 = 2 \sigma dl_1 \sin \beta_1 + 2 \sigma dl_2 \sin \beta_2 \quad (1.44)$$

For a very small area, the angles are very small and thus $(\sin \beta \sim \beta)$. Furthermore, it can be noticed that $dl_i \sim 2 R_i d\beta_i$. Thus, the equation (1.44) can be simplified as

$$\Delta P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (1.45)$$

Equation (1.45) predicts that pressure difference increase with inverse of the radius. There are two extreme cases: one) radius of infinite and radius of finite size. The second with two equal radii. The first case is for an infinite long cylinder for which the equation (1.45) is reduced to

$$\Delta P = \sigma \left(\frac{1}{R} \right) \quad (1.46)$$

Other extreme is for a sphere for which the main radii are the same and equation (1.45) is reduced to

$$\Delta P = \frac{2\sigma}{R} \quad (1.47)$$

Where R is the radius of the sphere. A soap bubble is made of two layers, inner and outer, thus the pressure inside the bubble is

$$\Delta P = \frac{4\sigma}{R} \quad (1.48)$$

Example 1.16:

A glass tube is inserted into bath of mercury. It was observed that contact angle between the glass and mercury is 55° C.

1.7. SURFACE TENSION

The inner diameter is 0.02[m] and the outer diameter is 0.021[m]. Estimate the force due to the surface tension (tube is depicted in Figure 1.18). It can be assumed that the contact angle is the same for the inside and outside part of the tube. Estimate the depression size. Assume that the surface tension for this combination of material is 0.5 [N/m]

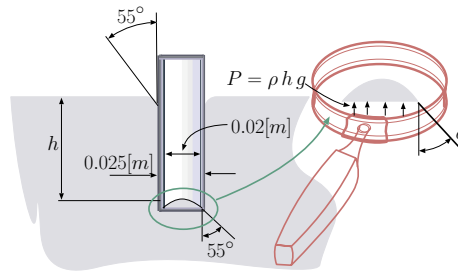


Fig. -1.18. Glass tube inserted into mercury.

SOLUTION

The mercury as free body that several forces act on it.

$$F = \sigma 2 \pi \cos 55^\circ C (D_i + D_o) \tag{1.XVI.a}$$

This force is upward and the horizontal force almost canceled. However, if the inside and the outside diameters are considerable different the results is

$$F = \sigma 2 \pi \sin 55^\circ C (D_o - D_i) \tag{1.XVI.b}$$

The balance of the forces on the meniscus show under the magnified glass are

$$P \overbrace{\pi r^2}^A = \sigma 2 \pi r + W \sim 0 \tag{1.XVI.c}$$

or

$$g \rho h \pi r^2 = \sigma 2 \pi r + W \sim 0 \tag{1.XVI.d}$$

Or after simplification

$$h = \frac{2 \sigma}{g \rho r} \tag{1.XVI.e}$$

End Solution

Example 1.17:

A Tank filled with liquid, which contains n bubbles with equal radii, r . Calculate the minimum work required to increase the pressure in tank by ΔP . Assume that the liquid bulk modulus is infinity.

SOLUTION

The work is due to the change of the bubbles volume. The work is

$$w = \int_{r_0}^{r_f} \Delta P(v) dv \tag{1.49}$$

The minimum work will be for a reversible process. The reversible process requires very slow compression. It is worth noting that for very slow process, the temperature must remain constant due to heat transfer. The relationship between pressure difference and the radius is described by equation (1.47) for reversible process. Hence the work is

$$w = \int_{r_0}^{r_f} \underbrace{\frac{\Delta P}{r}}_{\frac{2\sigma}{r}} \underbrace{4\pi r^2 dr}_{dv} = 8\pi\sigma \int_{r_0}^{r_f} r dr = 4\pi\sigma (r_f^2 - r_0^2) \quad (1.50)$$

Where, r_0 is the radius at the initial stage and r_f is the radius at the final stage.

The work for n bubbles is then $4\pi\sigma n (r_f^2 - r_0^2)$. It can be noticed that the work is negative, that is the work is done on the system.

End Solution

Example 1.18:

Calculate the rise of liquid between two dimensional parallel plates shown in Figure 1.19. Notice that previously a rise for circular tube was developed which different from simple one dimensional case. The distance between the two plates is ℓ and the surface tension is σ . Assume that the contact angle is 0^{circ} (the maximum possible force). Compute the value for surface tension of $0.05[N/m]$, the density $1000[kg/m^3]$ and distance between the plates of $0.001[m]$.

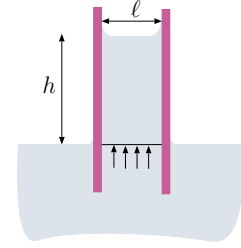


Fig. -1.19. Capillary rise between two plates.

SOLUTION

In Figure 1.19 exhibits the liquid under the current study. The vertical forces acting on the body are the gravity, the pressure above and below and surface tension. It can be noted that the pressure and above are the same with the exception of the curvature on the upper part. Thus, the control volume is taken just above the liquid and the air part is neglected. The question when the curvature should be answered in the Dimensional analysis and for simplification this effect is neglected. The net forces in the vertical direction (positive upwards) per unit length are

$$2\sigma \cos 0^\circ = g h \ell \rho \implies h = \frac{2\sigma}{\ell \rho g} \quad (1.51)$$

Inserting the values into equation (1.51) results in

$$h = \frac{2 \times 0.05}{0.001 \times 9.8 \times 1000} = \quad (1.52)$$

End Solution

Example 1.19:

Develop expression for rise of the liquid due to surface tension in concentric cylinders.

SOLUTION

The difference lie in the fact that “missing” cylinder add additional force and reduce the amount of liquid that has to raise. The balance between gravity and surface tension is

$$\sigma 2\pi (r_i \cos \theta_i + r_o \cos \theta_o) = \rho g h (\pi(r_o)^2 - \pi(r_i)^2) \quad (1.XIX.a)$$

Which can be simplified as

$$h = \frac{2\sigma (r_i \cos \theta_i + r_o \cos \theta_o)}{\rho g ((r_o)^2 - (r_i)^2)} \quad (1.XIX.b)$$

The maximum is obtained when $\cos \theta_i = \cos \theta_o = 1$. Thus, equation (1.XIX.b) can be simplified

$$h = \frac{2\sigma}{\rho g (r_o - r_i)} \quad (1.XIX.c)$$

End Solution

1.7.1 Wetting of Surfaces

To explain the source of the contact angle, consider the point where three phases became in contact. This contact point occurs due to free surface reaching a solid boundary. The surface tension occurs between gas phase (G) to liquid phase (L) and also occurs between the solid (S) and the liquid phases as well as between the gas phase and the solid phase. In Figure 1.20, forces diagram is shown when control volume is chosen so that the masses of the solid, liquid, and gas can be ignored. Regardless to the magnitude of the surface tensions (except to zero) the forces cannot be balanced for the description of straight lines. For example, forces balanced along the line of solid boundary is

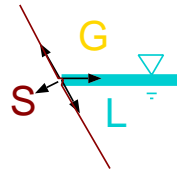


Fig. -1.20. Forces in Contact angle.

$$\sigma_{gs} - \sigma_{ls} - \sigma_{lg} \cos \beta = 0 \quad (1.53)$$

and in the tangent direction to the solid line the forces balance is

$$F_{solid} = \sigma_{lg} \sin \beta \quad (1.54)$$

substituting equation (1.54) into equation (1.53) yields

$$\sigma_{gs} - \sigma_{ls} = \frac{F_{solid}}{\tan \beta} \quad (1.55)$$

For $\beta = \pi/2 \implies \tan \beta = \infty$. Thus, the solid reaction force must be zero. The gas solid surface tension is different from the liquid solid surface tension and hence violating equation (1.53).

The surface tension forces must be balanced, thus, a contact angle is created to balance it. The contact angle is determined by whether the surface tension between the gas solid (gs) is larger or smaller than the surface tension of liquid solid (ls) and the local geometry. It must be noted that the solid boundary isn't straight. The surface tension is a molecular phenomenon, thus depend on the locale structure of the surface and it provides the balance for these local structures.

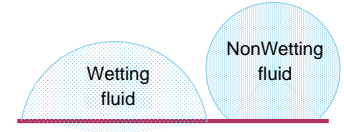


Fig. -1.21. Description of wetting and non-wetting fluids.

The connection of the three phases—materials—mediums creates two situations which are categorized as wetting or non-wetting. There is a common definition of wetting the surface. If the angle of the contact between three materials is larger than 90° then it is non-wetting. On the other hand, if the angle is below than 90° the material is wetting the surface (see Figure 1.21). The angle is determined by properties of the liquid, gas medium and the solid surface. And a small change on the solid surface can change the wetting condition to non-wetting. In fact there are commercial sprays that are intent to change the surface from wetting to non wetting. This fact is the reason that no reliable data can be provided with the exception to pure substances and perfect geometries. For example, water is described in many books as a wetting fluid. This statement is correct in most cases, however, when solid surface is made or cotted with certain materials, the water is changed to be wetting (for example 3M selling product to “change” water to non-wetting). So, the wetness of fluids is a function of the solid as well.

Table -1.6. The contact angle for air, distilled water with selected materials to demonstrate the inconsistency.

Chemical component	Contact Angle	Source
Steel	$\pi/3.7$	[1]
Steel,Nickel	$\pi/4.74$	[2]
Nickel	$\pi/4.74$ to $\pi/3.83$	[1]
Nickel	$\pi/4.76$ to $\pi/3.83$	[3]
Chrome-Nickel Steel	$\pi/3.7$	[4]
Continued on next page		

Table -1.6. The contact angle for air, distilled water with selected materials to demonstrate the inconsistency. (continue)

Chemical component	Contact Angle $\frac{mN}{m}$	Source
Silver	$\pi/6$ to $\pi/4.5$	[5]
Zink	$\pi/3.4$	[4]
Bronze	$\pi/3.2$	[4]
Copper	$\pi/4$	[4]
Copper	$\pi/3$	[7]
Copper	$\pi/2$	[8]

- 1 R. Siegel, E. G. Keshock (1975) "Effects of reduced gravity on nucleate boiling bubble dynamics in saturated water," AIChE Journal Volume 10 Issue 4, Pages 509 - 517. 1975
- 2 Bergles A. E. and Rohsenow W. M. "The determination of forced convection surface-boiling heat transfer, ASME, J. Heat Transfer, vol 1 pp 365 - 372.
- 3 Tolubinsky, V.I. and Ostrovsky, Y.N. (1966) "On the mechanism of boiling heat transfer", International Journal of Heat and Mass Transfer, Vol. 9, No 12, pages 1465-1470.
- 4 Arefeva E.I., Aladev O, I.T., (1958) "wlijanii smatchivaemosti na teploobmen pri kipenii," Inženerno Fizičeskij Jurnal, 11-17 1(7) In Russian.
- 5 Labuntsov D. A. (1939) "Approximate theory of heat transfer by developed nucleate boiling" In Sussian Izvestiya An SSSR , Energetika I transport, No 1.
- 6 Basu, N., Warriar, G. R., and Dhir, V. K., (2002) "Onset of Nucleate Boiling and Active Nucleation Site Density during Subcooled Flow Boiling," ASME Journal of Heat Transfer, Vol. 124, pages 717 -728.
- 7 Gaetner, R. F., and Westwater, J. W., (1960) "Population of Active Sites in Nucleate Boiling Heat Transfer," Chem. Eng. Prog. Symp., Ser. 56.
- 8 Wang, C. H., and Dhir, V. K., (1993), "Effect of Surface Wettability on Active Nucleation Site Density During Pool Boiling of Water on a Vertical Surface," J. Heat Transfer 115, pp. 659-669

To explain the contour of the surface, and the contact angle consider simple "wetting" liquid contacting a solid material in two-dimensional shape as depicted in Figure 1.22. To solve the shape of the liquid surface, the pressure difference between the two sides of free surface has to be balanced by the surface tension. In Figure 1.22 describes the raising of the liquid as results of the surface tension. The surface tension

reduces the pressure in the liquid above the liquid line (the dotted line in the Figure 1.22). The pressure just below the surface is $-g h(x) \rho$ (this pressure difference will be explained in more details in Chapter 4). The pressure, on the gas side, is the atmospheric pressure. This problem is a two dimensional problem and equation (1.46) is applicable to it. Applying equation (1.46) and using the pressure difference yields

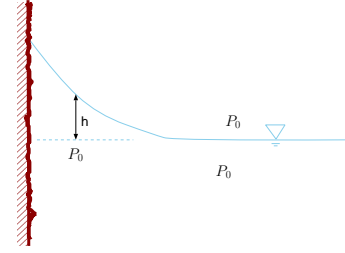


Fig. -1.22. Description of the liquid surface.

$$g h(x) \rho = \frac{\sigma}{R(x)} \quad (1.56)$$

The radius of any continuous function, $h = h(x)$, is

$$R(x) = \frac{\left(1 + [\dot{h}(x)]^2\right)^{3/2}}{\ddot{h}(x)} \quad (1.57)$$

Where \dot{h} is the derivative of h with respect to x .

Equation (1.57) can be derived either by forcing a circle at three points at $(x, x+dx, \text{ and } x+2dx)$ and thus finding the the diameter or by geometrical analysis of triangles build on points x and $x+dx$ (perpendicular to the tangent at these points). Substituting equation (1.57) into equation (1.56) yields

$$g h(x) \rho = \frac{\sigma}{\frac{\left(1 + [\dot{h}(x)]^2\right)^{3/2}}{\ddot{h}(x)}} \quad (1.58)$$

Equation (1.58) is non-linear differential equation for height and can be written as

1-D Surface Due to Surface Tension

$$\frac{g h \rho}{\sigma} \left(1 + \left[\frac{dh}{dx}\right]^2\right)^{3/2} - \frac{d^2 h}{dx^2} = 0 \quad (1.59)$$

With the boundary conditions that specify either the derivative $\dot{h}(x = r) = 0$ (symmetry) and the derivative at $\dot{h}x = \beta$ or heights in two points or other combinations. An alternative presentation of equation (1.58) is

$$g h \rho = \frac{\sigma \ddot{h}}{\left(1 + \dot{h}^2\right)^{3/2}} \quad (1.60)$$

Integrating equation (1.60) transforms into

$$\int \frac{g\rho}{\sigma} h dh = \int \frac{\ddot{h}}{(1 + \dot{h}^2)^{3/2}} dh \quad (1.61)$$

The constant $Lp\sigma/\rho g$ is referred to as Laplace's capillarity constant. The units of this constant are meter squared. The differential dh is \dot{h} . Using dummy variable and the identities $\dot{h} = \xi$ and hence, $\ddot{h} = d\xi$ transforms equation (1.61) into

$$\int \frac{1}{Lp} h dh = \int \frac{\xi d\xi}{(1 + \xi^2)^{3/2}} \quad (1.62)$$

After the integration equation (1.62) becomes

$$\frac{h^2}{2Lp} + constant = -\frac{1}{(1 + \dot{h}^2)^{1/2}} \quad (1.63)$$

At infinity, the height and the derivative of the height must be zero so $constant + 0 = -1/1$ and hence, $constant = -1$.

$$1 - \frac{h^2}{2Lp} = \frac{1}{(1 + \dot{h}^2)^{1/2}} \quad (1.64)$$

Equation (1.64) is a first order differential equation that can be solved by variables separation¹². Equation (1.64) can be rearranged to be

$$(1 + \dot{h}^2)^{1/2} = \frac{1}{1 - \frac{h^2}{2Lp}} \quad (1.65)$$

Squaring both sides and moving the one to the right side yields

$$\dot{h}^2 = \left(\frac{1}{1 - \frac{h^2}{2Lp}} \right)^2 - 1 \quad (1.66)$$

The last stage of the separation is taking the square root of both sides to be

$$\dot{h} = \frac{dh}{dx} = \sqrt{\left(\frac{1}{1 - \frac{h^2}{2Lp}} \right)^2 - 1} \quad (1.67)$$

¹²This equation has an analytical solution which is $x = Lp\sqrt{4 - (h/Lp)^2} - Lp \operatorname{acosh}(2Lp/h) + constant$ where Lp is the Laplace constant. Shamefully, this author doesn't know how to show it in a two lines derivations.

or

$$\frac{dh}{\sqrt{\left(\frac{1}{1 - \frac{h^2}{2Lp}}\right)^2 - 1}} = dx \quad (1.68)$$

Equation (1.68) can be integrated to yield

$$\int \frac{dh}{\sqrt{\left(\frac{1}{1 - \frac{h^2}{2Lp}}\right)^2 - 1}} = x + \text{constant} \quad (1.69)$$

The constant is determined by the boundary condition at $x = 0$. For example if $h(x = 0) = h_0$ then $\text{constant} = h_0$. This equation is studied extensively in classes on surface tension. Furthermore, this equation describes the dimensionless parameter that affects this phenomenon and this parameter will be studied in Chapter ?. This book is introductory, therefore this discussion on surface tension equation will be limited.

1.7.1.1 Capillarity

The capillary forces referred to the fact that surface tension causes liquid to rise or penetrate into area (volume), otherwise it will not be there. It can be shown that the height that the liquid raised in a tube due to the surface tension is

$$h = \frac{2\sigma \cos \beta}{g \Delta \rho r} \quad (1.70)$$

Where $\Delta \rho$ is the difference of liquid density to the gas density and r is the radius of tube.

But this simplistic equation is unusable and useless unless the contact angle (assuming that the contact angle is constant or a repressive average can be found or provided or can be measured) is given. However, in reality there is no readily information for contact angle¹³ and therefore this equation is useful to show the trends. The maximum that the contact angle can be obtained in equation (1.70) when $\beta = 0$ and thus $\cos \beta = 1$. This angle is obtained when a perfect half a sphere shape exist of the liquid surface. In that case equation (1.70) becomes

$$h_{max} = \frac{2\sigma}{g \Delta \rho r} \quad (1.71)$$

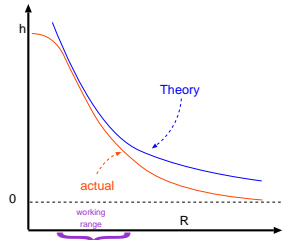


Fig. -1.23. The raising height as a function of the radii.

¹³Actually, there are information about the contact angle. However, that information conflict each other and no real information is available see Table 1.6.

Figure 1.24 exhibits the height as a function of the radius of the tube. The height based on equation (1.71) is shown in Figure 1.23 as blue line. The actual height is shown in the red line. Equation (1.71) provides reasonable results only in a certain range. For a small tube radius, equation (1.59) proved better results because the curve approaches hemispherical sphere (small gravity effect). For large radii equation (1.59) approaches the straight line (the liquid line) strong gravity effect. On the other hand, for extremely small radii equation (1.71) indicates that the high height which indicates a negative pressure. The liquid at a certain pressure will be vaporized and will breakdown the model upon this equation was constructed. Furthermore, the small scale indicates that the simplistic and continuous approach is not appropriate and a different model is needed. The conclusion of this discussion are shown in Figure 1.23. The actual dimension for many liquids (even water) is about 1-5 [mm]. The discussion above was referred to “wetting” contact angle. The depression of the liquid occurs in a “negative” contact angle similarly to “wetting.” The depression height, h is similar to equation (1.71) with a minus sign. However, the gravity is working against the surface tension and reducing the range and quality of the predictions of equation (1.71). The measurements of the height of distilled water and mercury are presented in Figure 1.24. The experimental results of these materials are with agreement with the discussion above.

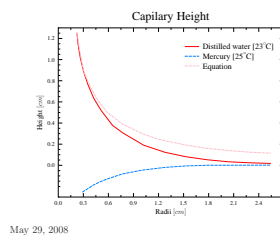


Fig. -1.24. The raising height as a function of the radius.

The surface tension of a selected material is given in Table 1.7.

In conclusion, the surface tension issue is important only in case where the radius is very small and gravity is negligible. The surface tension depends on the two materials or mediums that it separates.

Example 1.20:

Calculate the diameter of a water droplet to attain pressure difference of $1000[N/m^2]$. You can assume that temperature is $20^\circ C$.

SOLUTION

The pressure inside the droplet is given by equation (1.47).

$$D = 2R = \frac{2\sigma}{\Delta P} = \frac{4 \times 0.0728}{1000} \sim 2.912 \cdot 10^{-4} [m]$$

End Solution

Example 1.21:

Calculate the pressure difference between a droplet of water at $20^\circ C$ when the droplet has a diameter of 0.02 cm.

SOLUTION

using equation

$$\Delta P = \frac{2\sigma}{r} \sim \frac{2 \times 0.0728}{0.0002} \sim 728.0 [N/m^2]$$

End Solution

Example 1.22:

Calculate the maximum force necessary to lift a thin wire ring of 0.04[m] diameter from a water surface at 20°C. Neglect the weight of the ring.

SOLUTION

$$F = 2(2\pi r \sigma) \cos \beta$$

The actual force is unknown since the contact angle is unknown. However, the maximum Force is obtained when $\beta = 0$ and thus $\cos \beta = 1$. Therefore,

$$F = 4\pi r \sigma = 4 \times \pi \times 0.04 \times 0.0728 \sim .0366 [N]$$

In this value the gravity is not accounted for.

End Solution

Example 1.23:

A small liquid drop is surrounded with the air and has a diameter of 0.001 [m]. the pressure difference between the inside and outside droplet is 1[kPa]. Estimate the surface tension?

SOLUTION

To be continue

End Solution

Table -1.7. The surface tension for selected materials at temperature 20°C when not mentioned.

Chemical component	Surface Tension $\frac{mN}{m}$	T	correction $\frac{mN}{mK}$
Acetic Acid	27.6	20°C	n/a
Acetone	25.20	-	-0.1120
Aniline	43.4	22°C	-0.1085
Benzene	28.88	-	-0.1291
Benzylalcohol	39.00	-	-0.0920

Continued on next page

Table -1.7. The surface tension for selected materials (continue)

Chemical component	Surface Tension $\frac{mN}{m}$	T	correction $\frac{mN}{m K}$
Benzylbenzoate	45.95	-	-0.1066
Bromobenzene	36.50	-	-0.1160
Bromobenzene	36.50	-	-0.1160
Bromoform	41.50	-	-0.1308
Butyronitrile	28.10	-	-0.1037
Carbon disulfid	32.30	-	-0.1484
Quinoline	43.12	-	-0.1063
Chloro benzene	33.60	-	-0.1191
Chloroform	27.50	-	-0.1295
Cyclohexane	24.95	-	-0.1211
Cyclohexanol	34.40	25°C	-0.0966
Cyclopentanol	32.70	-	-0.1011
Carbon Tetrachloride	26.8	-	n/a
Carbon disulfid	32.30	-	-0.1484
Chlorobutane	23.10	-	-0.1117
Ethyl Alcohol	22.3	-	n/a
Ethanol	22.10	-	-0.0832
Ethylbenzene	29.20	-	-0.1094
Ethylbromide	24.20	-	-0.1159
Ethylene glycol	47.70	-	-0.0890
Formamide	58.20	-	-0.0842
Gasoline	~ 21	-	n/a
Glycerol	64.0	-	-0.0598
Helium	0.12	-269°C	n/a
Mercury	425-465.0	-	-0.2049
Methanol	22.70	-	-0.0773
Methyl naphthalene	38.60	-	-0.1118
Methyl Alcohol	22.6	-	n/a
Neon	5.15	-247°C	n/a
Nitrobenzene	43.90	-	-0.1177
Olive Oil	43.0-48.0	-	-0.067
Perfluoroheptane	12.85	-	-0.0972
Perfluorohexane	11.91	-	-0.0935
Perfluorooctane	14.00	-	-0.0902
Phenylisothiocyanate	41.50	-	-0.1172
Propanol	23.70	25°C	-0.0777
Pyridine	38.00	-	-0.1372

Continued on next page

Table -1.7. The surface tension for selected materials (continue)

Chemical component	Surface Tension $\frac{mN}{m}$	T	correction $\frac{mN}{m K}$
Pyrrrol	36.60	-	-0.1100
SAE 30 Oil	n/a	-	n/a
Seawater	54-69	-	n/a
Toluene	28.4	-	-0.1189
Turpentine	27	-	n/a
Water	72.80	-	-0.1514
o-Xylene	30.10	-	-0.1101
m-Xylene	28.90	-	-0.1104