

Note:

CHAPTER 5: MASS

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“Basics of Fluid Mechanics”

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Part I

Integral Analysis

CHAPTER 5

The Control Volume and Mass Conservation

5.1 Introduction

This chapter presents a discussion on the control volume and will be focused on the conservation of the mass. When the fluid system moves or changes, one wants to find or predict the velocities in the system. The main target of such analysis is to find the value of certain variables. This kind of analysis is reasonable and it referred to in the literature as the Lagrangian Analysis. This name is in honored J. L. Langrange (1736–1813) who formulated the equations of motion for the moving fluid particles.

Even though this system looks reasonable, the Lagrangian system turned out to be difficult to solve and to analyze. This method applied and used in very few cases. The main difficulty lies in the fact that every particle has to be traced to its original state. Leonard Euler (1707–1783) suggested an alternative approach. In Euler's approach the focus is on a defined point or a defined volume. This methods is referred as Eulerian method.

The Eulerian method focuses on a defined area or location to find the needed information. The use of the Eulerian methods leads to a set differentiation equations that is referred to as Navier–Stokes equations which are commonly used. These differential equations will be used in the later part of this book. Ad-

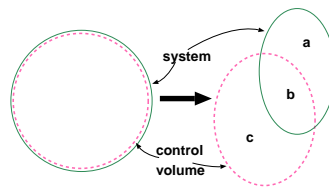


Fig. -5.1. Control volume and system before and after motion.

ditionally, the Eulerian system leads to integral equations which are the focus of this part of the book. The Eulerian method plays well with the physical intuition of most people. This method has its limitations and for some cases the Lagrangian is preferred (and sometimes the only possibility). Therefore a limited discussion on the Lagrangian system will be presented (later version).

Lagrangian equations are associated with the system while the Eulerian equation are associated with the control volume. The difference between the system and the control volume is shown in Figure ???. The green lines in Figure ??? represent the system. The red dotted lines are the control volume. At certain time the system and the control volume are identical location. After a certain time, some of the mass in the system exited the control volume which are marked "a" in Figure ???. The material that remained in the control volume is marked as "b". At the same time, the control volume gains some material which is marked as "c".

5.2 Control Volume

The Eulerian method requires to define a control volume (some time more than one). The control volume is a defined volume that was discussed earlier. The control volume is differentiated into two categories of control volumes, non-deformable and deformable.

Non-deformable control volume is a control volume which is fixed in space relatively to an one coordinate system. This coordinate system may be in a relative motion to another (almost absolute) coordinate system.

Deformable control volume is a volume having part of all of its boundaries in motion during the process at hand.

In the case where no mass crosses the boundaries, the control volume is a system. Every control volume is the focus of the certain interest and will be dealt with the basic equations, mass, momentum, energy, entropy etc.

Two examples of control volume are presented to illustrate difference between a deformable control volume and non-deformable control volume. Flow in conduits can be analyzed by looking in a control volume between two locations. The coordinate system could be fixed to the conduit. The control volume chosen is non-deformable control volume. The control volume should be chosen so that the analysis should be simple and dealt with as less as possible issues which are not in question. When a piston pushing gases a good choice of control volume is a deformable control volume that is a head the piston inside the cylinder as shown in Figure 5.2.

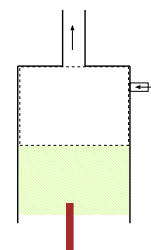


Fig. -5.2. Control volume of a moving piston with in and out flow.

5.3 Continuity Equation

In this chapter and the next three chapters, the conservation equations will be applied to the control volume. In this chapter, the mass conservation will be discussed. The system mass change is

$$\frac{D m_{sys}}{Dt} = \frac{D}{Dt} \int_{V_{sys}} \rho dV = 0 \tag{5.1}$$

The system mass after some time, according to Figure ??, is made of

$$m_{sys} = m_{c.v.} + m_a - m_c \tag{5.2}$$

The change of the system mass is by definition is zero. The change with time (time derivative of equation (5.2)) results in

$$0 = \frac{D m_{sys}}{Dt} = \frac{d m_{c.v.}}{dt} + \frac{d m_a}{dt} - \frac{d m_c}{dt} \tag{5.3}$$

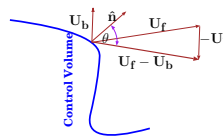
The first term in equation (5.3) is the derivative of the mass in the control volume and at any given time is

$$\frac{d m_{c.v.}(t)}{dt} = \frac{d}{dt} \int_{V_{c.v.}} \rho dV \tag{5.4}$$

and is a function of the time.

The interface of the control volume can move. The actual velocity of the fluid leaving the control volume is the relative velocity (see Figure 5.3). The relative velocity is

$$\vec{U}_r = \vec{U}_f - \vec{U}_b \tag{5.5}$$



Where U_f is the liquid velocity and U_b is the boundary velocity (see Figure 5.3). The velocity component that is perpendicular to the surface is

Fig. -5.3. Schematics of velocities at the interface.

$$U_{rn} = -\hat{n} \cdot \vec{U}_r = U_r \cos \theta \tag{5.6}$$

Where \hat{n} is an unit vector perpendicular to the surface. The convention of direction is taken positive if flow out the control volume and negative if the flow is into the control volume. The mass flow out of the control volume is the system mass that is not included in the control volume. Thus, the flow out is

$$\frac{d m_a}{dt} = \int_{S_{cv}} \rho_s U_{rn} dA \tag{5.7}$$

It has to be emphasized that the density is taken at the surface thus the subscript s . In the same manner, the flow rate in is

$$\frac{dm_b}{dt} = \int_{S_{c.v.}} \rho_s U_{rn} dA \quad (5.8)$$

It can be noticed that the two equations (5.8) and (5.7) are similar and can be combined, taking the positive or negative value of U_{rn} with integration of the entire system as

$$\frac{dm_a}{dt} - \frac{dm_b}{dt} = \int_{S_{c.v.}} \rho_s U_{rn} dA \quad (5.9)$$

applying negative value to keep the convention. Substituting equation (5.9) into equation (5.3) results in

$$\frac{d}{dt} \int_{c.v.} \rho_s dV = - \int_{S_{c.v.}} \rho U_{rn} dA \quad (5.10)$$

Equation (5.10) is essentially accounting of the mass. Again notice the negative sign in surface integral. The negative sign is because flow out marked positive which reduces of the mass (negative derivative) in the control volume. The change of mass change inside the control volume is net flow in or out of the control system.

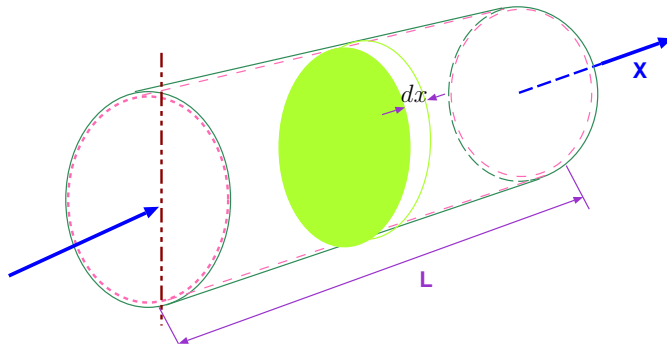


Fig. -5.4. Schematics of flow in in pipe with varying density as a function time for example 5.1.

The next example is provided to illustrate this concept.

Example 5.1:

The density changes in a pipe, due to temperature variation and other reasons, can be approximated as

$$\frac{\rho(x, t)}{\rho_0} = \left(1 - \frac{x}{L}\right)^2 \cos \frac{t}{t_0}.$$

The conduit shown in Figure 5.4 length is L and its area is A . Express the mass flow in and/or out, and the mass in the conduit as function of time. Write the expression for the mass change in the pipe.

SOLUTION

Here it is very convenient to choose a non-deformable control volume that is inside the conduit dV is chosen as $\pi R^2 dx$. Using equation (5.10), the flow out (or in) is

$$\frac{d}{dt} \int_{c.v.} \rho dV = \frac{d}{dt} \int_{c.v.} \overbrace{\rho_0 \left(1 - \frac{x}{L}\right)^2 \cos\left(\frac{t}{t_0}\right)}^{\rho(t)} \overbrace{\pi R^2 dx}^{dV}$$

The density is not a function of radius, r and angle, θ and they can be taken out the integral as

$$\frac{d}{dt} \int_{c.v.} \rho dV = \pi R^2 \frac{d}{dt} \int_{c.v.} \rho_0 \left(1 - \frac{x}{L}\right)^2 \cos\left(\frac{t}{t_0}\right) dx$$

which results in

$$\text{Flow Out} = \overbrace{\pi R^2}^A \frac{d}{dt} \int_0^L \rho_0 \left(1 - \frac{x}{L}\right)^2 \cos \frac{t}{t_0} dx = -\frac{\pi R^2 L \rho_0}{3 t_0} \sin\left(\frac{t}{t_0}\right)$$

The flow out is a function of length, L , and time, t , and is the change of the mass in the control volume.

End Solution

5.3.1 Non Deformable Control Volume

When the control volume is fixed with time, the derivative in equation (5.10) can enter the integral since the boundaries are fixed in time and hence,

$$\int_{V_{c.v.}} \frac{d\rho}{dt} dV = - \int_{S_{c.v.}} \rho U_{rn} dA \quad (5.11)$$

Equation (5.11) is simpler than equation (5.10).

5.3.2 Constant Density Fluids

Further simplifications of equations (5.10) can be obtained by assuming constant density and the equation (5.10) become conservation of the volume.

5.3.2.1 Non Deformable Control Volume

For this case the volume is constant therefore the mass is constant, and hence the mass change of the control volume is zero. Hence, the net flow (in and out) is zero. This condition can be written mathematically as

$$\overbrace{\frac{d}{dt} \int}^{=0} V_{rn} dA \rightarrow \int_{S_{c.v.}} V_{rn} dA = 0 \quad (5.12)$$

or in a more explicit form as

$$\int_{S_{in}} V_{rn} dA = \int_{S_{out}} V_{rn} dA = 0 \quad (5.13)$$

Notice that the density does not play a role in this equation since it is canceled out. Physically, the meaning is that volume flow rate in and the volume flow rate out have to equal.

5.3.2.2 Deformable Control Volume

The left hand side of question (5.10) can be examined further to develop a simpler equation by using the extend Leibniz integral rule for a constant density and result in

$$\frac{d}{dt} \int_{c.v.} \rho dV = \overbrace{\int_{c.v.} \frac{d\rho}{dt} dV}^{\text{thus, } =0} + \rho \int_{S_{c.v.}} \hat{n} \cdot U_b dA = \rho \int_{S_{c.v.}} U_{bn} dA \quad (5.14)$$

where U_b is the boundary velocity and U_{bn} is the normal component of the boundary velocity.

$$\int_{S_{c.v.}} U_{bn} dA = \int_{S_{c.v.}} U_{rn} dA \quad (5.15)$$

The meaning of the equation (5.15) is the net growth (or decrease) of the Control volume is by net volume flow into it. Example 5.2 illustrates this point.

Example 5.2:

Liquid fills a bucket as shown in Figure 5.5. The average velocity of the liquid at the exit of the filling pipe is U_p and cross section of the pipe is A_p . The liquid fills a bucket with cross section area of \mathbf{A} and instantaneous height is h . Find the height as a function of the other parameters. Assume that the density is constant and at the boundary interface $A_j = 0.7 A_p$. And where A_j is the area of jet when touching the liquid boundary in bucket. The last assumption is result of the energy equation (with

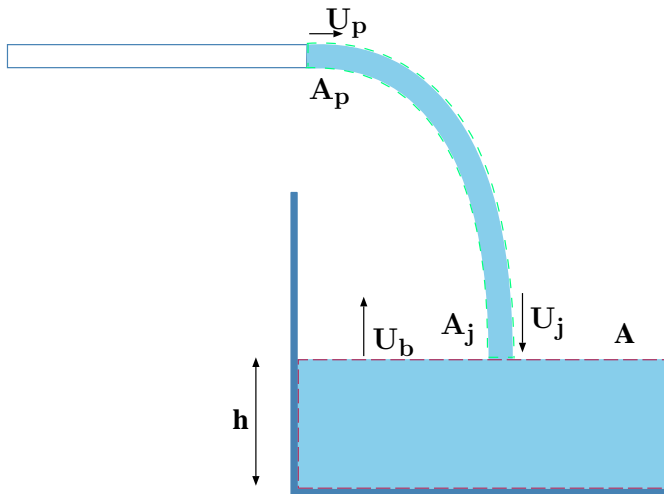


Fig. -5.5. Filling of the bucket and choices of the deformable control volumes for example 5.2.

some influence of momentum equation). The relationship is function of the distance of the pipe from the boundary of the liquid. However, this effect can be neglected for this range which this problem. In reality, the ratio is determined by height of the pipe from the liquid surface in the bucket. Calculate the bucket liquid interface velocity.

SOLUTION

This problem requires two deformable control volumes. The first control is around the jet and second is around the liquid in the bucket. In this analysis, several assumptions must be made. First, no liquid leaves the jet and enters the air. Second, the liquid in the bucket has a straight surface. This assumption is a strong assumption for certain conditions but it will be not discussed here since it is advance topic. Third, there are no evaporation or condensation processes. Fourth, the air effects are negligible. The control volume around the jet is deformable because the length of the jet shrinks with the time. The mass conservation of the liquid in the bucket is

$$\overbrace{\int_{c.v.} U_{bn} dA}^{\text{boundary change}} = \overbrace{\int_{c.v.} U_{rn} dA}^{\text{flow in}}$$

where U_{bn} is the perpendicular component of velocity of the boundary. Substituting the known values for U_{rn} results in

$$\int_{c.v.} U_b dA = \int_{c.v.} \overbrace{(U_j + U_b)}^{U_{rn}} dA$$

The integration can be carried when the area of jet is assumed to be known as

$$U_b A = A_j (U_j + U_b) \quad (5.11.a)$$

To find the jet velocity, U_j , the second control volume around the jet is used as the following

$$\overbrace{U_p A_p}^{\text{flow in}} - \overbrace{A_j (U_b + U_j)}^{\text{flow out}} = \overbrace{-A_j U_b}^{\text{boundary change}} \quad (5.11.b)$$

The above two equations (5.11.a) and (5.11.b) are enough to solve for the two unknowns. Substituting the first equation, (5.11.a) into (5.11.b) and using the ratio of $A_j = 0.7 A_p$ results

$$U_p A_p - U_b A = -0.7 A_p U_b \quad (5.11.c)$$

The solution of equation (5.11.c) is

$$U_b = \frac{A_p}{A - 0.7 A_p}$$

It is interesting that many individuals intuitively will suggest that the solution is $U_b A_p / A$. When examining solution there are two limits. The first limit is when $A_p = A / 0.7$ which is

$$U_b = \frac{A_p}{0} = \infty$$

The physical meaning is that surface is filled instantly. The other limit is that and $A_p / A \rightarrow 0$ then

$$U_b = \frac{A_p}{A}$$

which is the result for the "intuitive" solution. It also interesting to point out that if the filling was from other surface (not the top surface), e.g. the side, the velocity will be $U_b = U_p$ in the limiting case and not infinity. The reason for this difference is that the liquid already fill the bucket and has not to move into bucket.

End Solution

Example 5.3:

Balloon is attached to a rigid supply in which is supplied by a constant the mass rate, m_i . Calculate the velocity of the balloon boundaries assuming constant density.

SOLUTION

The applicable equation is

$$\int_{c.v.} U_{bn} dA = \int_{c.v.} U_{rn} dA$$

The entrance is fixed, thus the relative velocity, U_{rn} is

$$U_{rn} = \begin{cases} -U_p & \text{@ the valve} \\ 0 & \text{every else} \end{cases}$$

Assume equal distribution of the velocity in balloon surface and that the center of the balloon is moving, thus the velocity has the following form

$$U_b = U_x \hat{x} + U_{br} \hat{r}$$

Where \hat{x} is unit coordinate in x direction and U_x is the velocity of the center and where \hat{r} is unit coordinate in radius from the center of the balloon and U_{br} is the velocity in that direction. The right side of equation (5.15) is the net change due to the boundary is

$$\int_{S_{c.v.}} (U_x \hat{x} + U_{br} \hat{r}) \cdot \hat{n} dA = \overbrace{\int_{S_{c.v.}} (U_x \hat{x}) \cdot \hat{n} dA}^{\text{center movement}} + \overbrace{\int_{S_{c.v.}} (U_{br} \hat{r}) \cdot \hat{n} dA}^{\text{net boundary change}}$$

The first integral is zero because it is like movement of solid body and also yield this value mathematically (excises for mathematical oriented student). The second integral (notice $\hat{n} = \hat{r}$) yields

$$\int_{S_{c.v.}} (U_{br} \hat{r}) \cdot \hat{n} dA = 4 \pi r^2 U_{br}$$

Substituting into the general equation yields

$$\rho \overbrace{4 \pi r^2}^A U_{br} = \rho U_p A_p = m_i$$

Hence,

$$U_{br} = \frac{m_i}{\rho 4 \pi r^2}$$

The center velocity is (also) exactly U_{br} . The total velocity of boundary is

$$U_t = \frac{m_i}{\rho 4 \pi r^2} (\hat{x} + \hat{r})$$

It can be noticed that the velocity at the opposite to the connection to the rigid pipe which is double of the center velocity.

End Solution

5.3.2.3 One-Dimensional Control Volume

Additional simplification of the continuity equation is of one dimensional flow. This simplification provides very useful description for many fluid flow phenomena. The main assumption made in this model is that the properties in the across section are only function of x coordinate . This assumptions leads

$$\int_{A_2} \rho_2 U_2 dA - \int_{A_1} \rho_1 U_1 dA = \frac{d}{dt} \int_{V(x)} \rho(x) \overbrace{A(x) dx}^{dV} \quad (5.16)$$

When the density can be considered constant equation (5.16) is reduced to

$$\int_{A_2} U_2 dA - \int_{A_1} U_1 dA = \frac{d}{dt} \int A(x) dx \quad (5.17)$$

For steady state but with variations of the velocity and variation of the density reduces equation (5.16) to become

$$\int_{A_2} \rho_2 U_2 dA = \int_{A_1} \rho_1 U_1 dA \quad (5.18)$$

For steady state and uniform density and velocity equation (5.18) reduces further to

$$\rho_1 A_1 U_1 = \rho_2 A_2 U_2 \quad (5.19)$$

For incompressible flow (constant density), continuity equation is at its minimum form of

$$U_1 A_1 = A_2 U_2 \quad (5.20)$$

The next example is of semi one-dimensional example to illustrate equation (5.16).

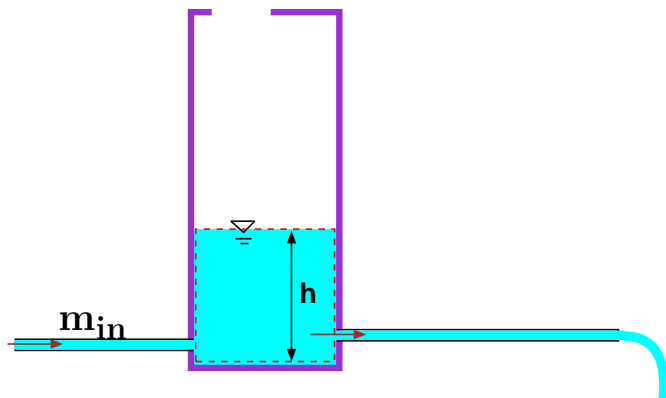


Fig. -5.6. Height of the liquid for example 5.4.

Example 5.4:

Liquid flows into tank in a constant mass flow rate of a . The mass flow rate out is function of the height. First assume that $q_{out} = b h$ second Assume as $q_{out} = b \sqrt{h}$. For the first case, determine the height, h as function of the time. Is there a critical value and then if exist find the critical value of the system parameters. Assume that the height at time zero is h_0 . What happen if the $h_0 = 0$?

SOLUTION

The control volume for both cases is the same and it is around the liquid in the tank. It can be noticed that control volume satisfy the demand of one dimensional since the flow is only function of x coordinate. For case one the right hand side term in equation (5.16) is

$$\rho \frac{d}{dt} \int_0^L h dx = \rho L \frac{dh}{dt}$$

Substituting into equation equation (5.16) is

$$\rho L \frac{dh}{dt} = \overbrace{b_1 h}^{\text{flow out}} - \overbrace{m_i}^{\text{flow in}}$$

solution is

$$h = \overbrace{\frac{m_i}{b_1} e^{-\frac{b_1 t}{\rho L}}}^{\text{homogeneous solution}} + \overbrace{c_1 e^{\frac{b_1 t}{\rho L}}}^{\text{private solution}}$$

The solution has the homogeneous solution (solution without the m_i) and the solution of the m_i part. The solution can rearranged to a new form (a discussion why this form is preferred will be provided in dimensional chapter).

$$\frac{h b_1}{m_1} = e^{-\frac{b_1 t}{\rho L}} + c e^{\frac{b_1 t}{\rho L}}$$

With the initial condition that at $h(t=0) = h_0$ the constant coefficient can be found as

$$\frac{h_0 b_1}{m_1} = 1 - c \implies c = 1 - \frac{h_0 b_1}{m_i}$$

which the solution is

$$\frac{h b_1}{m_1} = e^{-\frac{b_1 t}{\rho L}} + \left[1 - \frac{h_0 b_1}{m_i} \right] e^{\frac{b_1 t}{\rho L}}$$

It can be observed that if $1 = \frac{h_0 b_1}{m_i}$ is the critical point of this solution. If the term $\frac{h_0 b_1}{m_i}$ is larger than one then the solution reduced to a negative number. However, negative number for height is not possible and the height solution approach zero. If the reverse case appeared, the height will increase. Essentially, the critical ratio state if the flow in is larger or lower than the flow out determine the condition of the height.

For second case, the governing equation (5.16) is

$$\rho L \frac{dh}{dt} = \overbrace{b \sqrt{h}}^{\text{flow out}} - \overbrace{m_i}^{\text{flow in}}$$

with the general solution of

$$\ln \left[\left(\frac{\sqrt{h} b}{m_i} - 1 \right) \frac{m_i}{\rho L} \right] + \frac{\sqrt{h} b}{m_i} - 1 = (t + c) \frac{\sqrt{h} b}{2 \rho L}$$

The constant is obtained when the initial condition that at $h(t = 0) = h_0$ and it left as exercise for the reader.

End Solution

5.4 Reynolds Transport Theorem

It can be noticed that the same derivations carried for the density can be carried for other intensive properties such as specific entropy, specific enthalpy. Suppose that g is intensive property (which can be a scalar or a vector) undergoes change with time. The change of accumulative property will be then

$$\frac{D}{Dt} \int_{sys} f \rho dV = \frac{d}{dt} \int_{c.v.} f \rho dV + \int_{c.v.} f \rho U_{rn} dA \quad (5.21)$$

This theorem named after Reynolds, Osborne, (1842-1912) which is actually a three dimensional generalization of Leibniz integral rule¹. To make the previous derivation clearer, the Reynolds Transport Theorem will be reproofed and discussed. The ideas are the similar but extended some what.

Leibniz integral rule² is an one dimensional and it is defined as

$$\frac{d}{dy} \int_{x_1(y)}^{x_2(y)} f(x, y) dx = \int_{x_1(y)}^{x_2(y)} \frac{\partial f}{\partial y} dx + f(x_2, y) \frac{dx_2}{dy} - f(x_1, y) \frac{dx_1}{dy} \quad (5.22)$$

Initially, a proof will be provided and the physical meaning will be explained. Assume that there is a function that satisfy the following

$$G(x, y) = \int^x f(\alpha, y) d\alpha \quad (5.23)$$

Notice that lower boundary of the integral is missing and is only the upper limit of the function is present³. For its derivative of equation (5.23) is

$$f(x, y) = \frac{\partial G}{\partial x} \quad (5.24)$$

differentiating (chain rule $d w = u dv + v du$) by part of left hand side of the Leibniz integral rule (it can be shown which are identical) is

$$\frac{d [G(x_2, y) - G(x_1, y)]}{dy} = \overbrace{\frac{\partial G}{\partial x_2} \frac{dx_2}{dy}}^1 + \overbrace{\frac{\partial G}{\partial y}(x_2, y)}^2 - \overbrace{\frac{\partial G}{\partial x_1} \frac{dx_1}{dy}}^3 - \overbrace{\frac{\partial G}{\partial y}(x_1, y)}^4 \quad (5.25)$$

¹These papers can be read on-line at <http://www.archive.org/details/papersonmechanic01reynrich>.

²This material is not necessarily but is added her for completeness. This author find material just given so no questions will be asked.

³There was a suggestion to insert arbitrary constant which will be canceled and will a provide rigorous proof. This is engineering book and thus, the exact mathematical proof is not the concern here. Nevertheless, if there will be a demand for such, it will be provided.

The terms 2 and 4 in equation (5.25) are actually (the x_2 is treated as a different variable)

$$\int_{x_1(y)}^{x_2(y)} \frac{\partial f(x, y)}{\partial y} dx \quad (5.26)$$

The first term (1) in equation (5.25) is

$$\frac{\partial G}{\partial x_2} \frac{dx_2}{dy} = f(x_2, y) \frac{dx_2}{dy} \quad (5.27)$$

The same can be said for the third term (3). Thus this explanation is a proof the Leibniz rule.

The above “proof” is mathematical in nature and physical explanation is also provided. Suppose that a fluid is flowing in a conduit. The intensive property, f is investigated or the accumulative property, F . The interesting information that commonly needed is the change of the accumulative property, F , with time. The change with time is

$$\frac{DF}{Dt} = \frac{D}{Dt} \int_{sys} \rho f dV \quad (5.28)$$

For one dimensional situation the change with time is

$$\frac{DF}{Dt} = \frac{D}{Dt} \int_{sys} \rho f A(x) dx \quad (5.29)$$

If two limiting points (for the one dimensional) are moving with a different coordinate system, the mass will be different and it will not be a system. This limiting condition is the control volume for which some of the mass will leave or enter. Since the change is very short (differential), the flow in (or out) will be the velocity of fluid minus the boundary at x_1 , $U_{rn} = U_1 - U_b$. The same can be said for the other side. The accumulative flow of the property in, F , is then

$$F_{in} = \underbrace{f_1}_{F_1} \rho \underbrace{U_{rn}}_{\frac{dx_1}{dt}} \quad (5.30)$$

The accumulative flow of the property out, F , is then

$$F_{out} = \underbrace{f_2}_{F_2} \rho \underbrace{U_{rn}}_{\frac{dx_2}{dt}} \quad (5.31)$$

The change with time of the accumulative property, F , between the boundaries is

$$\frac{d}{dt} \int_{c.v.} \rho(x) f A(x) dA \quad (5.32)$$

When put together it brings back the Leibniz integral rule. Since the time variable, t , is arbitrary and it can be replaced by any letter. The above discussion is one of the physical meaning the Leibniz rule.

Reynolds Transport theorem is a generalization of the Leibniz rule and thus the same arguments are used. The only difference is that the velocity has three components and only the perpendicular component enters into the calculations.

$$\frac{D}{DT} \int_{sys} f \rho dV = \frac{d}{dt} \int_{c.v} f \rho dV + \int_{S_{c.v.}} f \rho U_{rn} dA \quad (5.33)$$

5.5 Examples For Mass Conservation

Several examples are provided to illustrate the topic.

Example 5.5:

Liquid enters a circular pipe with a linear velocity profile as a function of the radius with maximum velocity of U_{max} . After magical mixing, the velocity became uniform. Write the equation which describes the velocity at the entrance. What is the magical averaged velocity at the exit? Assume no-slip condition.

SOLUTION

The velocity profile is linear with radius. Additionally, later a discussion on relationship between velocity at interface to solid also referred as the (no) slip condition will be provided. This assumption is good for most cases with very few exceptions. It will be assumed that the velocity at the interface is zero. Thus, the boundary condition is $U(r = R) = 0$ and $U(r = 0) = U_{max}$. Therefore the velocity profile is

$$U(r) = U_{max} \left(1 - \frac{r}{R}\right)$$

Where R is radius and r is the working radius (for the integration). The magical averaged velocity is obtained using the equation (5.13). For which

$$\int_0^R U_{max} \left(1 - \frac{r}{R}\right) 2\pi r dr = U_{ave} \pi R^2 \quad (5.V.a)$$

The integration of the equation (5.V.a) is

$$U_{max} \pi \frac{R^2}{6} = U_{ave} \pi R^2 \quad (5.V.b)$$

The solution of equation (b) results in average velocity as

$$U_{ave} = \frac{U_{max}}{6} \quad (5.V.c)$$

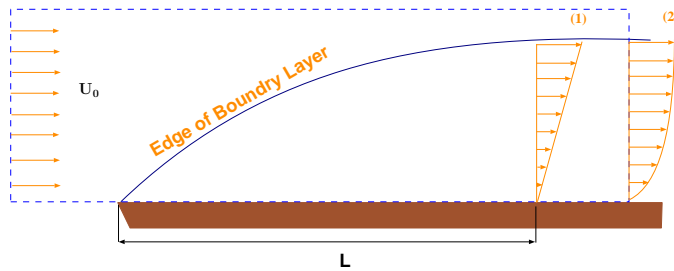


Fig. -5.7. Boundary Layer control mass.

Example 5.6:

Experiments have shown that a layer of liquid that attached itself to the surface and it is referred to as boundary layer. The assumption is that fluid attaches itself to surface. The slowed liquid is slowing the layer above it. The boundary layer is growing with x because the boundary effect is penetrating further into fluid. A common boundary layer analysis uses the Reynolds transform theorem. In this case, calculate the relationship of the mass transfer across the control volume. For simplicity assume slowed fluid has a linear velocity profile. Then assume parabolic velocity profile as

$$U_x(y) = 2U_0 \left[\frac{y}{\delta} + \frac{1}{2} \left(\frac{y}{\delta} \right)^2 \right]$$

and calculate the mass transfer across the control volume. Compare the two different velocity profiles affecting on the mass transfer.

SOLUTION

Assuming the velocity profile is linear thus, (to satisfy the boundary condition) it will be

$$U_x(y) = \frac{U_0 y}{\delta}$$

The chosen control volume is rectangular of $L \times \delta$. Where δ is the height of the boundary layer at exit point of the flow as shown in Figure 5.7. The control volume has three surfaces that mass can cross, the left, right, and upper. No mass can cross the lower surface (solid boundary). The situation is steady state and thus using equation (5.13) results in

$$\overbrace{\int_0^\delta U_0 dy - \int_0^\delta \frac{U_0 y}{\delta} dy}^{\text{x direction}} = \overbrace{\int_0^L U_x dx}^{\text{y direction}}$$

It can be noticed that the convention used in this chapter of "in" as negative is not "followed." The integral simply multiply by negative one. The above integrals on the

right hand side can be combined as

$$\int_0^\delta U_0 \left(1 - \frac{y}{\delta}\right) dy = \int_0^L U x dx$$

the integration results in

$$\frac{U_0 \delta}{2} = \int_0^L U x dx$$

or for parabolic profile

$$\int_0^\delta U_0 dy - \int_0^\delta U_0 \left[\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2\right] dy = \int_0^L U x dx$$

or

$$\int_0^\delta U_0 \left[1 - \frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] dy = U_0$$

the integration results in

$$\frac{U_0 \delta}{2} = \int_0^L U x dx$$

End Solution

Example 5.7:

Air flows into a jet engine at 5 kg/sec while fuel flow into the jet is at 0.1 kg/sec. The burned gases leaves at the exhaust which has cross area 0.1 m² with velocity of 500 m/sec. What is the density of the gases at the exhaust?

SOLUTION

The mass conservation equation (5.13) is used. Thus, the flow out is (5 + 0.1) 5.1 kg/sec The density is

$$\rho = \frac{\dot{m}}{AU} = \frac{5.1 \text{ kg/sec}}{0.01 \text{ m}^2 \cdot 500 \text{ m/sec}} = 1.02 \text{ kg/m}^3$$

End Solution

The mass (volume) flow rate is given by direct quantity like $x \text{ kg/sec}$. However sometime, the mass (or the volume) is given by indirect quantity such as the effect of flow. The next example deal with such reversed mass flow rate.

Example 5.8:

The tank is filled by two valves which one filled tank in 3 hours and the second by 6 hours. The tank also has three emptying valves of 5 hours, 7 hours, and 8 hours. The tank is 3/4 fulls, calculate the time for tank reach empty or full state when all the valves are open. Is there a combination of valves that make the tank at steady state?

SOLUTION

Easier measurement of valve flow rate can be expressed as fraction of the tank per hour. For example valve of 3 hours can be converted to $1/3$ tank per hour. Thus, mass flow rate in is

$$\dot{m}_{in} = 1/3 + 1/6 = 1/2 \text{ tank/hour}$$

The mass flow rate out is

$$\dot{m}_{out} = 1/5 + 1/7 + 1/8 = \frac{131}{280}$$

Thus, if all the valves are open the tank will be filled. The time to completely filled the tank is

$$\frac{\frac{1}{4}}{\frac{1}{2} - \frac{131}{280}} = \frac{70}{159} \text{ hour}$$

The rest is under construction.

End Solution

Example 5.9:

Inflated cylinder is supplied in its center with constant mass flow. Assume that the gas mass is supplied in uniform way of m_i [kg/m/sec]. Assume that the cylinder inflated uniformly and pressure inside the cylinder is uniform. The gas inside the cylinder obeys the ideal gas law. The pressure inside the cylinder is linearly proportional to the volume. For simplicity, assume that the process is isothermal. Calculate the cylinder boundaries velocity.

SOLUTION

The applicable equation is

$$\overbrace{\int_{V_{c.v.}} \frac{d\rho}{dt} dV}^{\text{increase pressure}} + \overbrace{\int_{S_{c.v.}} \rho U_b dV}^{\text{boundary velocity}} = \overbrace{\int_{S_{c.v.}} \rho U_{rn} dA}^{\text{in or out flow rate}}$$

Every term in the above equation is analyzed but first the equation of state and volume to pressure relationship have to be provided.

$$\rho = \frac{P}{RT}$$

and relationship between the volume and pressure is

$$P = f \pi R_c^2$$

Where R_c is the instantaneous cylinder radius. Combining the above two equations results in

$$\rho = \frac{f \pi R_c^2}{RT}$$

Where f is a coefficient with the right dimension. It also can be noticed that boundary velocity is related to the radius in the following form

$$U_b = \frac{dR_c}{dt}$$

The first term requires to find the derivative of density with respect to time which is

$$\frac{d\rho}{dt} = \frac{d}{dt} \left(\frac{f \pi R_c^2}{RT} \right) = \frac{2 f \pi R_c}{RT} \overbrace{\frac{dR_c}{dt}}^{U_b}$$

Thus the first term is

$$\int_{V_{c.v.}} \frac{d\rho}{dt} \overbrace{dV}^{2\pi R_c} = \int_{V_{c.v.}} \frac{2 f \pi R_c}{RT} U_b \overbrace{dV}^{2\pi R_c} = \frac{4 f \pi^2 R_c^3}{3 RT} U_b$$

The integral can be carried when U_b is independent of the R_c^4 The second term is

$$\int_{S_{c.v.}} \rho U_b dA = \overbrace{\frac{f \pi R_c^2}{RT}}^{\rho} U_b \overbrace{2\pi R_c}^A = \left(\frac{f \pi^3 R_c^2}{RT} \right) U_b$$

substituting in the governing equation obtained the form of

$$\frac{f \pi^2 R_c^3}{RT} U_b + \frac{4 f \pi^2 R_c^3}{3 RT} U_b = m_i$$

The boundary velocity is then

$$U_b = \frac{m_i}{\frac{7 f \pi^2 R_c^3}{3 RT}} = \frac{3 m_i RT}{7 f \pi^2 R_c^3}$$

End Solution

Example 5.10:

A balloon is attached to a rigid supply and is supplied by a constant mass rate, m_i . Assume that gas obeys the ideal gas law. Assume that balloon volume is a linear function of the pressure inside the balloon such as $P = f_v V$. Where f_v is a coefficient describing the balloon physical characters. Calculate the velocity of the balloon boundaries under the assumption of isothermal process.

⁴The proof of this idea is based on the chain differentiation similar to Leibniz rule. When the derivative of the second part is $dU_b/dR_c = 0$.

SOLUTION

The question is more complicated than Example 5.10. The ideal gas law is

$$\rho = \frac{P}{RT}$$

The relationship between the pressure and volume is

$$P = f_v V = \frac{4 f_v \pi R_b^3}{3}$$

The combining of the ideal gas law with the relationship between the pressure and volume results

$$\rho = \frac{4 f_v \pi R_b^3}{3 RT}$$

The applicable equation is

$$\int_{V_{c.v.}} \frac{d\rho}{dt} dV + \int_{S_{c.v.}} \rho (U_c \hat{x} + U_b \hat{r}) dA = \int_{S_{c.v.}} \rho U_{rn} dA$$

The right hand side of the above equation is

$$\int_{S_{c.v.}} \rho U_{rn} dA = m_i$$

The density change is

$$\frac{d\rho}{dt} = \frac{12 f_v \pi R_b^2}{RT} \overbrace{\frac{dR_b}{dt}}^{U_b}$$

The first term is

$$\int_0^{R_b} \overbrace{\frac{12 f_v \pi R_b^2}{RT}}^{\neq f(r)} U_b \overbrace{4 \pi r^2 dr}^{dV} = \frac{16 f_v \pi^2 R_b^5}{3 RT} U_b$$

The second term is

$$\int_A \frac{4 f_v \pi R_b^3}{3 RT} U_b dA = \frac{4 f_v \pi R_b^3}{3 RT} U_b \overbrace{4 \pi R_b^2}^A = \frac{8 f_v \pi^2 R_b^5}{3 RT} U_b$$

Substituting the two equations of the applicable equation results

$$U_b = \frac{1}{8} \frac{m_i RT}{f_v \pi^2 R_b^5}$$

Notice that first term is used to increase the pressure and second the change of the boundary.

Open Question: Answer must be received by April 15, 2010

The best solution of the following question will win 18 U.S. dollars and your name will be associated with the solution in this book.

Example 5.11:

Solve example 5.10 under the assumption that the process is isentropic. Also assume that the relationship between the pressure and the volume is $P = f_v V^2$. What are the units of the coefficient f_v in this problem? What are the units of the coefficient in the previous problem?

5.6 The Details Picture – Velocity Area Relationship

The integral approach is intended to deal with the “big” picture. Indeed the method is used in this part of the book for this purpose. However, there is very little written about the usability of this approach to provide way to calculate the average quantities in the control system. Sometime is desirable to find the averaged velocity or velocity distribution inside a control volume. There is no general way to provide these quantities. Therefore an example will be provided to demonstrate the use of this approach.

Consider a container filled with liquid on which one exit opened and the liquid flows out as shown in Figure 5.8. The velocity has three components in each of the coordinates under the assume that flow is uniform and the surface is straight⁵. The integral approached is used to calculate the averaged velocity of each to the components. To relate the velocity in the z direction with the flow rate out or the exit the velocity mass balance is constructed. A similar control volume construction to find the velocity of the boundary velocity (height) can be carried out. The control volume is bounded by the container wall including the exit of the flow. The upper boundary is surface parallel to upper surface but at Z distance from the bottom. The mass balance reads

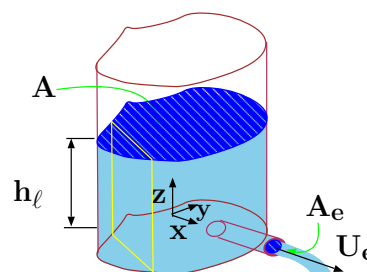


Fig. -5.8. Control volume usage to calculate local averaged velocity in Three coordinates.

$$\int_V \frac{d\rho}{dt} dV + \int_A U_{bn} \rho dA + \int_A U_{rn} \rho dA = 0 \quad (5.34)$$

⁵The liquid surface is not straight for this kind of problem. However, under certain conditions it is reasonable to assume straight surface which have been done for this problem.

For constant density (conservation of volume) equation⁶ and ($h > z$) reduces to

$$\int_A U_{rn} \rho dA = 0 \tag{5.35}$$

In the container case for uniform velocity equation 5.35 becomes

$$U_z A = U_e A_e \implies U_z = -\frac{A_e}{A} U_e \tag{5.36}$$

It can be noticed that the boundary is not moving and the mass inside does not change inside this control volume. The velocity U_z is the averaged velocity downward.

The x component of velocity is obtained by using a different control volume. The control volume is shown in Figure 5.9. The boundary are the container far from the flow exit with blue line projection into page (area) shown in the Figure 5.9. The mass conservation for constant density of this control volume is

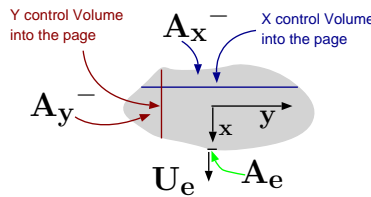


Fig. -5.9. Control volume and system before and after motion.

$$-\int_A U_{bn} \rho dA + \int_A U_{rn} \rho dA = 0 \tag{5.37}$$

Usage of control volume not included in the previous analysis provides the velocity at the upper boundary which is the same as the velocity at y direction. Substituting into (5.37) results in

$$\int_{A_{x^-}} \frac{A_e}{A} U_e \rho dA + \int_{A_{yz}} U_x \rho dA = 0 \tag{5.38}$$

Where A_{x^-} is the area shown the Figure under this label. The area A_{yz} referred to area into the page in Figure 5.9 under the blue line. Because averaged velocities and constant density are used transformed equation (5.38) into

$$\frac{A_e}{A} A_{x^-} U_e + U_x \overbrace{Y(x) h}^{A_{yz}} = 0 \tag{5.39}$$

Where $Y(x)$ is the length of the (blue) line of the boundary. It can be notice that the velocity, U_x is generally increasing with x because A_{x^-} increase with x .

The calculations for the y directions are similar to the one done for x direction. The only difference is that the velocity has two different directions. One zone is right to the exit with flow to the left and one zone to left with averaged velocity to right. If the volumes on the left and the right are symmetrical the averaged velocity will be zero.

⁶The point where ($z = h$) the boundary term is substituted the flow in term.

Example 5.12:

Calculate the velocity, U_x for a cross section of circular shape (cylinder).

SOLUTION

The relationship for this geometry needed to be expressed. The length of the line $Y(x)$ is

$$Y(x) = 2r \sqrt{1 - \left(1 - \frac{x}{r}\right)^2} \quad (5.XII.a)$$

This relationship also can be expressed in the term of α as

$$Y(x) = 2r \sin \alpha \quad (5.XII.b)$$

Since this expression is simpler it will be adapted. When the relationship between radius angle and x are

$$x = r(1 - \sin \alpha) \quad (5.XII.c)$$

The area A_x^- is expressed in term of α as

$$A_x^- = \left(\alpha - \frac{1}{2} \sin(2\alpha)\right) r^2 \quad (5.XII.d)$$

Thus the velocity, U_x is

$$\frac{A_e}{A} \left(\alpha - \frac{1}{2} \sin(2\alpha)\right) r^2 U_e + U_x 2r \sin \alpha h = 0 \quad (5.XII.e)$$

$$U_x = \frac{A_e r}{A h} \frac{\left(\alpha - \frac{1}{2} \sin(2\alpha)\right)}{\sin \alpha} U_e \quad (5.XII.f)$$

Averaged velocity is defined as

$$\bar{U}_x = \frac{1}{S} \int_S U dS \quad (5.XII.g)$$

Where here S represent some length. The same way it can be represented for angle calculations. The value dS is $r \cos \alpha$. Integrating the velocity for the entire container and dividing by the angle, α provides the averaged velocity.

$$\bar{U}_x = \frac{1}{2r} \int_0^\pi \frac{A_e r}{A h} \frac{\left(\alpha - \frac{1}{2} \sin(2\alpha)\right)}{\tan \alpha} U_e r d\alpha \quad (5.XII.h)$$

which results in

$$\bar{U}_x = \frac{(\pi - 1)}{4} \frac{A_e r}{A h} U_e \quad (5.XII.i)$$

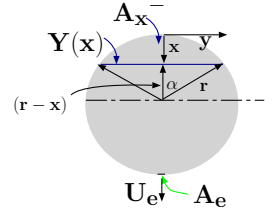


Fig. -5.10. Circular cross section for Finding U_x and various cross sections.

End Solution

Example 5.13:

Calculate the velocity, U_y for a cross section of circular shape (cylinder). What is the averaged velocity if only half section is used. State your assumptions and how it similar to the previous example.

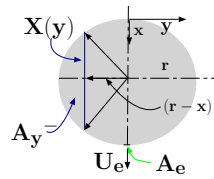


Fig. -5.11. y velocity for a circular shape

SOLUTION

The flow out in the x direction is zero because symmetrical reasons. That is the flow field is a mirror images. Thus, every point has different velocity with the same value in the opposite direction.

The flow in half of the cylinder either the right or the left has non zero averaged velocity. The calculations are similar to those in the previous to example 5.12. The main concept that must be recognized is the half of the flow must have come from one side and the other come from the other side. Thus, equation (5.39) modified to be

$$\frac{A_e}{A} A_x - U_e + U_x \overbrace{Y(x)}^{A_{yz}} h = 0 \tag{5.40}$$

The integral is the same as before but the upper limit is only to $\pi/2$

$$\bar{U}_x = \frac{1}{2r} \int_0^{\pi/2} \frac{A_e r}{A h} \frac{(\alpha - \frac{1}{2} \sin(2\alpha))}{\tan \alpha} U_e r d\alpha \tag{5.XIII.a}$$

which results in

$$\bar{U}_x = \frac{(\pi - 2)}{8} \frac{A_e r}{A h} U_e \tag{5.XIII.b}$$

End Solution

5.7 More Examples for Mass Conservation

Typical question about the relative velocity that appeared in many fluid mechanics exams is the following.

Example 5.14:

A boat travels at speed of $10m/sec$ upstream in a river that flows at a speed of $5m/s$. The inboard engine uses a pump to suck

in water at the front $A_{in} = 0.2 \text{ m}^2$ and eject it through the back of the boat with exist area of $A_{out} = 0.05 \text{ m}^2$. The water absolute velocity leaving the back is 50 m/sec , what are the relative velocities entering and leaving the boat and the pumping rate?

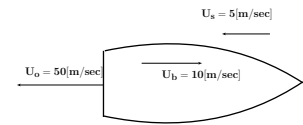


Fig. -5.12. Schematic of the boat for example 5.14

SOLUTION

The boat is assumed (implicitly is stated) to be steady state and the density is constant. However, the calculation have to be made in the frame of reference moving with the boat. The relative jet discharge velocity is

$$U_{r_{out}} = 50 - (10 + 5) = 35 [m/sec]$$

The volume flow rate is then

$$Q_{out} = A_{out} U_{r_{out}} = 35 \times 0.05 = 1.75 \text{ m}^3/sec$$

The flow rate at entrance is the same as the exit thus,

$$U_{r_{in}} = \frac{A_{out}}{A_{in}} U_{r_{out}} = \frac{0.05}{0.2} 35 = 8.75 \text{ m/sec}$$