

# Note:

CHAPTER 4: MECH

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This chapter is part of the textbook:

**“Basics of Fluid Mechanics”**

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DECEMBER 21, 2011

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NSY = Not Started Yet

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# CHAPTER 3

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## Review of Mechanics

*This author would like to express his gratitude to Dan Olsen (former Minneapolis city Engineer) and his friend Richard Hackbarth.*

This chapter provides a review of important definitions and concepts from Mechanics (statics and dynamics). These concepts and definitions will be used in this book and a review is needed.

### 3.1 Kinematics of of Point Body

A point body is location at time,  $t$  in a location,  $\vec{R}$ . The velocity is derivative of the change of the location and using the chain rule (for the direction and one for the magnitude) results,

$$\vec{U} = \frac{d\vec{R}}{dt} = \overbrace{\left. \frac{d\vec{R}}{dt} \right|_R}^{\text{change in } R \text{ direction}} + \overbrace{\vec{\omega} \times \vec{R}}^{\text{change in perpendicular to } R} \quad (3.1)$$

Notice that  $\vec{\omega}$  can have three dimensional components. It also can be noticed that this derivative is present derivation of any victory. The acceleration is the derivative of the velocity

$$\vec{a} = \frac{d\vec{U}}{dt} = \overbrace{\left. \frac{d^2\vec{R}}{dt^2} \right|_R}^{\text{"regular acceleration"}} + \overbrace{\left( \vec{R} \times \frac{d\vec{\omega}}{dt} \right)}^{\text{angular acceleration}} + \overbrace{\vec{\omega} \times \left( \vec{R} \times \vec{\omega} \right)}^{\text{centrifugal acceleration}} + 2 \overbrace{\left( \left. \frac{d\vec{R}}{dt} \right|_R \times \vec{\omega} \right)}^{\text{Coriolis acceleration}} \quad (3.2)$$

Example 3.1:

*A water jet is supposed be used to extinguish the fire in a building as depicted in Figure*

3.1<sup>1</sup>. For given velocity, at what angle the jet has to be shot so that velocity will be horizontal at the window. Assume that gravity is  $g$  and the distance of the nozzle from the building is  $a$  and height of the window from the nozzle is  $b$ . To simplify the calculations, it proposed to calculate the velocity of the point particle to toward the window. Calculate what is the velocity so that the jet reach the window. What is the angle that jet has to be aimed.

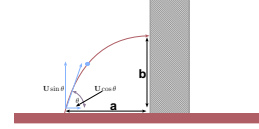


Fig. -3.1. Description of the extinguish nozzle aimed at the building window.

#### SOLUTION

The initial velocity is unknown and denoted as  $U$  which two components. The velocity at  $x$  is  $U_x = U \cos \theta$  and the velocity in  $y$  direction is  $U_y = U \sin \theta$ . There there are three unknowns,  $U$ ,  $\theta$ , and time,  $t$  and three equations. The equation for the  $x$  coordinate is

$$a = U \cos \theta t \quad (3.1.a)$$

The distance for  $y$  equation for coordinate (zero is at the window) is

$$0 = -\frac{gt^2}{2} + U \sin \theta t - b \quad (3.1.b)$$

The velocity for the  $y$  coordinate at the window is zero

$$u(t) = 0 = -gt + U \sin \theta \quad (3.1.c)$$

These nonlinear equations (3.1.a), (3.1.b) and (3.1.c) can be solved explicitly. Isolating  $t$  from (3.1.a) and substituting into equations (3.1.b) and (3.1.c)

$$b = \frac{-ga^2}{2U^2 \cos^2 \theta} + a \tan \theta \quad (3.1.d)$$

and equation (3.1.a) becomes

$$0 = \frac{-ga}{U \cos \theta} + U \cos \theta \implies U = \frac{\sqrt{ag}}{\cos \theta} \quad (3.1.e)$$

Substituting (3.1.e) into (3.1.d) results in

$$\tan \theta = \frac{b}{a} + \frac{1}{2} \quad (3.1.f)$$

End Solution

<sup>1</sup>While the simple example does not provide exact use of the above equation it provides experience of going over the motions of kinematics.

## 3.2 Center of Mass

The center of mass is divided into two sections, first, center of the mass and two, center of area (two-dimensional body with equal distribution mass).

### 3.2.1 Actual Center of Mass

In many engineering problems, the center of mass is required to make the calculations. This concept is derived from the fact that a body has a center of mass/gravity which interacts with other bodies and that this force acts on the center (equivalent force). It turns out that this concept is very useful in calculating rotations, moment of inertia, etc. The center of mass doesn't depend on the coordinate system and on the way it is calculated. The physical meaning of the center of mass is that if a straight line force acts on the body in away through the center of gravity, the body will not rotate. In other words, if a body will be held by one point it will be enough to hold the body in the direction of the center of mass. Note, if the body isn't be held through the center of mass, then a moment in addition to force is required (to prevent the body for rotating). It is convenient to use the Cartesian system to explain this concept. Suppose that the body has a distribution of the mass (density,  $\rho$ ) as a function of the location. The density "normally" defined as mass per volume. Here, the the line density is referred to density mass per unit length in the  $x$  direction.

In  $x$  coordinate, the center will be defined as

$$\bar{x} = \frac{1}{m} \int_V x \overbrace{\rho(x)dV}^{dm} \quad (3.3)$$

Here, the  $dV$  element has finite dimensions in  $y$ - $z$  plane and infinitesimal dimension in  $x$  direction see Figure 3.2. Also, the mass,  $m$  is the total mass of the object. It can be noticed that center of mass in the  $x$ -direction isn't affected by the distribution in the  $y$  nor by  $z$  directions. In same fashion the center of mass can be defined in the other directions as following

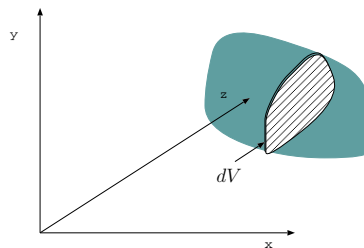


Fig. -3.2. Description of how the center of mass is calculated.

$$\bar{x}_i = \frac{1}{m} \int_V x_i \rho(x_i) dV \quad (3.4)$$

where  $x_i$  is the direction of either,  $x$ ,  $y$  or  $z$ . The density,  $\rho(x_i)$  is the line density as function of  $x_i$ . Thus, even for solid and uniform density the line density is a function of the geometry.

### 3.2.2 Approximate Center of Area

In the previous case, the body was a three dimensional shape. There are cases where the body can be approximated as a two-dimensional shape because the body is thin with uniform density. Consider a uniform thin body with constant thickness shown in Figure 3.3 which has density,  $\rho$ . Thus, equation (3.3) can be transferred into

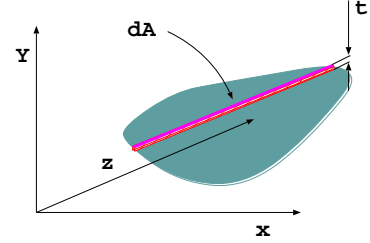


Fig. -3.3. Thin body center of mass/area schematic.

$$\bar{x} = \frac{1}{\underbrace{tA}_V} \rho \int_V x \overbrace{\rho t dA}^{dm} \quad (3.5)$$

The density,  $\rho$  and the thickness,  $t$ , are constant and can be canceled. Thus equation (3.5) can be transferred into

Approximate $x_i$ of Center Mass
$\bar{x}_i = \frac{1}{A} \int_A x_i dA \quad (3.6)$

when the integral now over only the area as oppose over the volume.

Finding the centroid location should be done in the most convenient coordinate system since the location is coordinate independent.

## 3.3 Moment of Inertia

As it was divided for the body center of mass, the moment of inertia is divided into moment of inertia of mass and area.

### 3.3.1 Moment of Inertia for Mass

The moment of inertia turns out to be an essential part for the calculations of rotating bodies. Furthermore, it turns out that the moment of inertia has much wider applicability. Moment of inertia of mass is defined as

Moment of Inertia
$I_{rrm} = \int_m \rho r^2 dm \quad (3.7)$

If the density is constant then equation (3.7) can be transformed into

$$I_{rrm} = \rho \int_V r^2 dV \quad (3.8)$$

The moment of inertia is independent of the coordinate system used for the calculation, but dependent on the location of axis of rotation relative to the body. Some people define the radius of gyration as an equivalent concepts for the center of mass concept and which means if all the mass were to locate in the one point/distance and to obtain the same of moment of inertia.

$$r_k = \sqrt{\frac{I_m}{m}} \tag{3.9}$$

The body has a different moment of inertia for every coordinate/axis and they are

$$\begin{aligned} I_{xx} &= \int_V r_x^2 dm = \int_V (y^2 + z^2) dm \\ I_{yy} &= \int_V r_y^2 dm = \int_V (x^2 + z^2) dm \\ I_{zz} &= \int_V r_z^2 dm = \int_V (x^2 + y^2) dm \end{aligned} \tag{3.10}$$

### 3.3.2 Moment of Inertia for Area

#### 3.3.2.1 General Discussion

For body with thickness,  $t$  and uniform density the following can be written

$$I_{xxm} = \int_m r^2 dm = \rho t \overbrace{\int_A r^2 dA}^{\text{moment of inertia for area}} \tag{3.11}$$

The moment of inertia about axis is  $x$  can be defined as

**Moment of Inertia**

$$I_{xx} = \int_A r^2 dA = \frac{I_{xxm}}{\rho t} \tag{3.12}$$

where  $r$  is distance of  $dA$  from the axis  $x$  and  $t$  is the thickness. Any point distance can be calculated from axis  $x$  as

$$r = \sqrt{y^2 + z^2} \tag{3.13}$$

Thus, equation (3.12) can be written as

$$I_{xx} = \int_A (y^2 + z^2) dA \tag{3.14}$$

In the same fashion for other two coordinates as

$$I_{yy} = \int_A (x^2 + z^2) dA \tag{3.15}$$

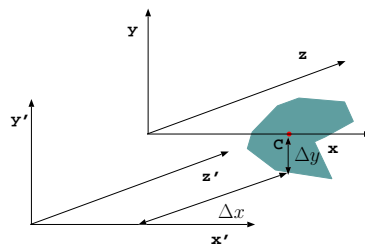


Fig. -3.4. The schematic that explains the summation of moment of inertia.

$$I_{zz} = \int_A (x^2 + y^2) dA \quad (3.16)$$

### 3.3.2.2 The Parallel Axis Theorem

The moment of inertia can be calculated for any axis. The knowledge about one axis can help calculating the moment of inertia for a parallel axis. Let  $I_{xx}$  the moment of inertia about axis  $xx$  which is at the center of mass/area.

The moment of inertia for axis  $x'$  is

$$I_{x'x'} = \int_A r'^2 dA = \int_A (y'^2 + z'^2) dA = \int_A [(y + \Delta y)^2 + (z + \Delta z)^2] dA \quad (3.17)$$

equation (3.17) can be expanded as

$$I_{x'x'} = \overbrace{\int_A (y^2 + z^2) dA}^{I_{xx}} + \overbrace{2 \int_A (y \Delta y + z \Delta z) dA}^{=0} + \int_A ((\Delta y)^2 + (\Delta z)^2) dA \quad (3.18)$$

The first term in equation (3.18) on the right hand side is the moment of inertia about axis  $x$  and the second term is zero. The second term is zero because it integral of center about center thus is zero. The third term is a new term and can be written as

$$\int_A \overbrace{((\Delta y)^2 + (\Delta z)^2)}^{\text{constant}} dA = \overbrace{((\Delta y)^2 + (\Delta z)^2)}^{r^2} \overbrace{\int_A dA}^A = r^2 A \quad (3.19)$$

Hence, the relationship between the moment of inertia at  $xx$  and parallel axis  $x'x'$  is

Parallel Axis Equation

$$I_{x'x'} = I_{xx} + r^2 A \quad (3.20)$$

The moment of inertia of several areas is the sum of moment inertia of each area see Figure 3.5 and therefore,

$$I_{xx} = \sum_{i=1}^n I_{xxi} \quad (3.21)$$

If the same areas are similar thus

$$I_{xx} = \sum_{i=1}^n I_{xxi} = n I_{xxi} \quad (3.22)$$

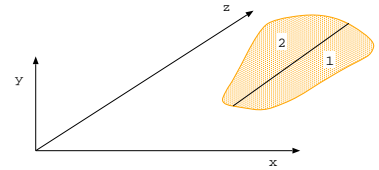


Fig. -3.5. The schematic to explain the summation of moment of inertia.

Equation (3.22) is very useful in the calculation of the moment of inertia utilizing the moment of inertia of known bodies. For example, the moment of inertial of half a circle is half of whole circle for axis a the center of circle. The moment of inertia can then move the center of area. of the

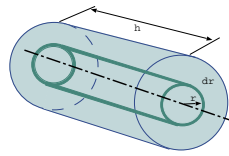


Fig. -3.6. Cylinder with an element for calculation moment of inertia.

### 3.3.3 Examples of Moment of Inertia

Example 3.2:

Calculate the moment of inertia for the mass of the cylinder about center axis which height of  $h$  and radius,  $r_0$ , as shown in Figure 3.6. The material is with an uniform density and homogeneous.

SOLUTION

The element can be calculated using cylindrical coordinate. Here the convenient element is a shell of thickness  $dr$  which shown in Figure 3.6 as

$$I_{rr} = \rho \int_V r^2 dm = \rho \int_0^{r_0} r^2 \overbrace{h 2 \pi r dr}^{dV} = \rho h 2 \pi \frac{r_0^4}{4} = \frac{1}{2} \rho h \pi r_0^4 = \frac{1}{2} m r_0^2$$

The radius of gyration is

$$r_k = \sqrt{\frac{\frac{1}{2} m r_0^2}{m}} = \frac{r_0}{\sqrt{2}}$$

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End Solution

Example 3.3:

Calculate the moment of inertia of the rectangular shape shown in Figure 3.7 around x coordinate.

SOLUTION

The moment of inertia is calculated utilizing equation (3.14) as following

$$I_{xx} = \int_A \left( \overbrace{y^2}^0 + z^2 \right) dA = \int_0^a z^2 \overbrace{bdz}^{dA} = \frac{a^3 b}{3}$$

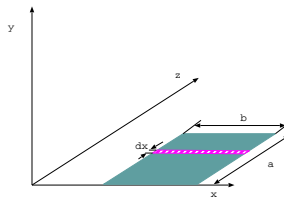


Fig. -3.7. Description of rectangular in x-y plane for calculation of moment of inertia.

This value will be used in later examples.

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End Solution

## Example 3.4:

To study the assumption of zero thickness, consider a simple shape to see the effects of this assumption. Calculate the moment of inertia about the center of mass of a square shape with a thickness,  $t$  compare the results to a square shape with zero thickness.

SOLUTION

The moment of inertia of transverse slice about  $y'$  (see Figure mech:fig:squareEll) is

$$dI_{xxm} = \rho \overbrace{t}^t \overbrace{ba^3}^{I_{xx}} \frac{1}{12} \quad (3.23)$$

The transformation into from local axis  $x$  to center axis,  $x'$  can be done as following

$$dI_{x'x'm} = \rho dy \left( \frac{\overbrace{ba^3}^{I_{xx}}}{12} + \underbrace{z^2}_{r^2} \underbrace{ba}_A \right) \quad (3.24)$$

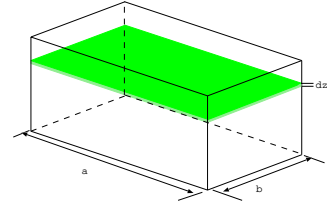


Fig. -3.8. A square element for the calculations of inertia of two-dimensional to three-dimensional deviations.

The total moment of inertia can be obtained by integration of equation (3.24) to write as

$$I_{xxm} = \rho \int_{-t/2}^{t/2} \left( \frac{ba^3}{12} + z^2 ba \right) dz = \rho t \frac{ab t^2 + a^3 b}{12} \quad (3.25)$$

Comparison with the thin body results in

$$\frac{I_{xx} \rho t}{I_{xxm}} = \frac{ba^3}{t^2 ba + ba^3} = \frac{1}{1 + \frac{t^2}{a^2}} \quad (3.26)$$

It can be noticed right away that equation (3.26) indicates that ratio approaches one when thickness ratio is approaches zero,  $I_{xxm}(t \rightarrow 0) \rightarrow 1$ . Additionally it can be noticed that the ratio  $a^2/t^2$  is the only contributor to the error<sup>2</sup>.

The results are present in Figure 3.9. I can be noticed that the error is significant very fast even for small values of  $t/a$  while the with of the box,  $b$  has no effect on the error.

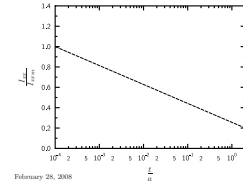


Fig. -3.9. The ratio of the moment of inertia of two-dimensional to three-dimensional.

End Solution

<sup>2</sup>This ratio is a dimensionless number that commonly has no special name. This author suggests to call this ratio as the B number.

Example 3.5:

Calculate the rectangular moment of Inertia for the rotation trough center in  $zz$  axis (axis of rotation is out of the page). Hint, construct a small element and build longer build out of the small one. Using this method calculate the entire rectangular.

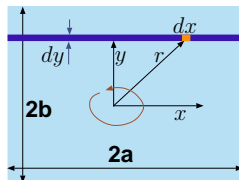


Fig. -3.10. Rectangular Moment of inertia.

SOLUTION

The moment of inertia for a long element with a distance  $y$  shown in Figure 3.10 is

$$dI_{zz}|_{dy} = \int_{-a}^a \overbrace{(y^2 + x^2)}^{r^2} dy dx = \frac{2(3ay^2 + a^3)}{3} dy \quad (3.V.a)$$

The second integration ( no need to use (3.20), why?) is

$$I_{zz} = \int_{-b}^b \frac{2(3ay^2 + a^3)}{3} dy \quad (3.V.b)$$

Results in

$$I_{zz} = \frac{a(2ab^3 + 2a^3b)}{3} = \overbrace{\frac{4ab}{A}}^A \left( \frac{(2a)^2 + (2b)^2}{12} \right) \quad (3.V.c)$$

Or

————— End Solution —————

Example 3.6:

Calculate the center of area and moment of inertia for the parabola,  $y = \alpha x^2$ , depicted in Figure 3.11. Hint, calculate the area first. Use this area to calculate moment of inertia. There are several ways to approach the calculation (different infinitesimal area).

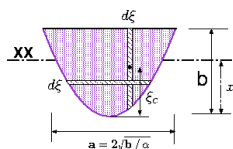


Fig. -3.11. Parabola for calculations of moment of inertia.

SOLUTION

For  $y = b$  the value of  $x = \sqrt{b/\alpha}$ . First the area inside the parabola calculated as

$$A = 2 \int_0^{\sqrt{b/\alpha}} \overbrace{(b - \alpha\xi^2)}^{dA/2} d\xi = \frac{2(3\alpha - 1)}{3} \left( \frac{b}{\alpha} \right)^{3/2}$$

The center of area can be calculated utilizing equation (3.6). The center of every element is at,  $\left(\alpha \xi^2 + \frac{b-\alpha \xi^2}{2}\right)$  the element area is used before and therefore

$$x_c = \frac{1}{A} \int_0^{\sqrt{b/\alpha}} \overbrace{\left(\alpha \xi^2 + \frac{b-\alpha \xi^2}{2}\right)}^{x_c} \overbrace{(b-\alpha \xi^2) d\xi}^{dA} = \frac{3\alpha b}{15\alpha - 5} \quad (3.27)$$

The moment of inertia of the area about the center can be found using in equation (3.27) can be done in two steps first calculate the moment of inertia in this coordinate system and then move the coordinate system to center. Utilizing equation (3.14) and doing the integration from 0 to maximum  $y$  provides

$$I_{x'x'} = 4 \int_0^b \xi^2 \overbrace{\sqrt{\frac{\xi}{\alpha}} d\xi}^{dA} = \frac{2b^{7/2}}{7\sqrt{\alpha}}$$

Utilizing equation (3.20)

$$I_{xx} = I_{x'x'} - A \Delta x^2 = \frac{4b^{7/2}}{7\sqrt{\alpha}} - \frac{3\alpha - 1}{3} \left(\frac{b}{\alpha}\right)^{3/2} \left(\frac{3\alpha b}{15\alpha - 5}\right)^2$$

or after working the details results in

$$I_{xx} = \frac{\sqrt{b} (20b^3 - 14b^2)}{35\sqrt{\alpha}}$$

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End Solution

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### Example 3.7:

Calculate the moment of inertia of straight angle triangle about its  $y$  axis as shown in the Figure on the right. Assume that base is  $a$  and the height is  $h$ . What is the moment when a symmetrical triangle is attached on left. What is the moment when a symmetrical triangle is attached on bottom. What is the moment inertia when  $a \rightarrow 0$ . What is the moment inertia when  $h \rightarrow 0$ .

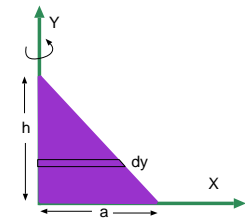


Fig. -3.12. Triangle for example 3.7.

### SOLUTION

The right edge line equation can be calculated as

$$\frac{y}{h} = \left(1 - \frac{x}{a}\right)$$

or

$$\frac{x}{a} = \left(1 - \frac{y}{h}\right)$$

Now using the moment of inertia of rectangle on the side ( $y$ ) coordinate (see example (3.3))

$$\int_0^h \frac{a \left(1 - \frac{y}{h}\right)^3 dy}{3} = \frac{a^3 h}{4}$$

For two triangles attached to each other the moment of inertia will be sum as  $\frac{a^3 h}{2}$   
The rest is under construction.

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End Solution

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### 3.3.4 Product of Inertia

In addition to the moment of inertia, the product of inertia is commonly used. Here only the product of the area is defined and discussed. The product of inertia defined as

$$I_{x_i x_j} = \int_A x_i x_j dA \quad (3.28)$$

For example, the product of inertia for  $x$  and  $y$  axes is

$$I_{xy} = \int_A x y dA \quad (3.29)$$

Product of inertia can be positive or negative value as oppose the moment of inertia. The calculation of the product of inertia isn't different much for the calculation of the moment of inertia. The units of the product of inertia are the same as for moment of inertia.

#### Transfer of Axis Theorem

Same as for moment of inertia there is also similar theorem.

$$I_{x' y'} = \int_A x' y' dA = \int_A (x + \Delta x)(y + \Delta y) dA \quad (3.30)$$

expanding equation (3.30) results in

$$I_{x' y'} = \underbrace{\int_A x y dA}_{I_{xy}} + \underbrace{\Delta y \int_A x dA}_{\Delta y \int_A x dA} + \underbrace{\Delta x \int_A y dA}_{\Delta x \int_A y dA} + \underbrace{\Delta x \Delta y A}_{\Delta x \Delta y A} \quad (3.31)$$

The final form is

$$I_{x'y'} = I_{xy} + \Delta x \Delta y A \quad (3.32)$$

There are several relationships should be mentioned

$$I_{xy} = I_{yx} \quad (3.33)$$

Symmetrical area has zero product of inertia because integration of odd function (asymmetrical function) left part cancel the right part.

Example 3.8:

Calculate the product of inertia of straight edge triangle.

SOLUTION

The equation of the line is

$$y = \frac{a}{b}x + a$$

The product of inertia at the center is zero. The total product of inertia is

$$I_{x'y'} = 0 + \underbrace{\frac{\Delta x}{3}}_a \underbrace{\frac{\Delta y}{3}}_b \underbrace{\left(\frac{ab}{2}\right)}_A = \frac{a^2 b^2}{18}$$

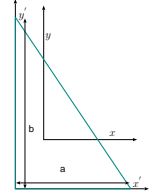


Fig. -3.13. Product of inertia for triangle.

End Solution

### 3.3.5 Principal Axes of Inertia

The inertia matrix or inertia tensor is

$$\begin{vmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{vmatrix} \quad (3.34)$$

In linear algebra it was shown that for some angle equation (3.34) can be transform into

$$\begin{vmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{vmatrix} \quad (3.35)$$

System which creates equation (3.35) referred as principle system.

### 3.4 Newton's Laws of Motion

These laws can be summarized in two statements one, for every action by body **A** on Body **B** there is opposite reaction by body **B** on body **A**. Two, which can expressed in mathematical form as

$$\sum \mathbf{F} = \frac{D(mU)}{Dt} \quad (3.36)$$

It can be noted that  $D$  replaces the traditional  $d$  since the additional meaning which be added. Yet, it can be treated as the regular derivative. This law apply to any body and any body can "broken" into many small bodies which connected to each other. These small "bodies" when became small enough equation (3.36) can be transformed to a continuous form as

$$\sum \mathbf{F} = \int_V \frac{D(\rho U)}{Dt} dV \quad (3.37)$$

The external forces are equal to internal forces the forces between the "small" bodies are cancel each other. Yet this examination provides a tool to study what happened in the fluid during operation of the forces.

Since the derivative with respect to time is independent of the volume, the derivative can be taken out of the integral and the alternative form can be written as

$$\sum \mathbf{F} = \frac{D}{Dt} \int_V \rho U dV \quad (3.38)$$

The velocity,  $U$  is a derivative of the location with respect to time, thus,

$$\sum \mathbf{F} = \frac{D^2}{Dt^2} \int_V \rho r dV \quad (3.39)$$

where  $r$  is the location of the particles from the origin.

The external forces are typically divided into two categories: body forces and surface forces. The body forces are forces that act from a distance like magnetic field or gravity. The surface forces are forces that act on the surface of the body (pressure, stresses). The same as in the dynamic class, the system acceleration called the internal forces. The acceleration is divided into three categories: Centrifugal,  $\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega})$ , Angular,  $\mathbf{r} \times \dot{\boldsymbol{\omega}}$ , Coriolis,  $2(\mathbf{U}_r \times \boldsymbol{\omega})$ . The radial velocity is denoted as  $U_r$ .

### 3.5 Angular Momentum and Torque

The angular momentum of body,  $dm$ , is defined as

$$L = \mathbf{r} \times \mathbf{U} dm \quad (3.40)$$

The angular momentum of the entire system is calculated by integration (summation) of all the particles in the system as

$$L_s = \int_m \mathbf{r} \times \mathbf{U} dm \quad (3.41)$$

The change with time of angular momentum is called torque, in analogous to the momentum change of time which is the force.

$$T_\tau = \frac{DL}{Dt} = \frac{D}{Dt} (\mathbf{r} \times \mathbf{U} dm) \quad (3.42)$$

where  $T_\tau$  is the torque. The torque of entire system is

$$T_{\tau_s} = \int_m \frac{DL}{Dt} = \frac{D}{Dt} \int_m (\mathbf{r} \times \mathbf{U} dm) \quad (3.43)$$

It can be noticed (well, it can be proved utilizing vector mechanics) that

$$T_\tau = \frac{D}{Dt} (\mathbf{r} \times \mathbf{U}) = \frac{D}{Dt} (\mathbf{r} \times \frac{D\mathbf{r}}{Dt}) = \frac{D^2\mathbf{r}}{Dt^2} \quad (3.44)$$

To understand these equations a bit better, consider a particle moving in x-y plane. A force is acting on the particle in the same plane (x-y) plane. The velocity can be written as  $U = u\hat{i} + v\hat{j}$  and the location from the origin can be written as  $\mathbf{r} = x\hat{i} + y\hat{j}$ . The force can be written, in the same fashion, as  $\mathbf{F} = F_x\hat{i} + F_y\hat{j}$ . Utilizing equation (3.40) provides

$$\mathbf{L} = \mathbf{r} \times \mathbf{U} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ u & v & 0 \end{pmatrix} = (xv - yu)\hat{k} \quad (3.45)$$

Utilizing equation (3.42) to calculate the torque as

$$T_\tau = \mathbf{r} \times \mathbf{F} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{pmatrix} = (xF_x - yF_y)\hat{k} \quad (3.46)$$

Since the torque is a derivative with respect to the time of the angular momentum it is also can be written as

$$xF_x - yF_y = \frac{D}{Dt} [(xv - yu) dm] \quad (3.47)$$

The torque is a vector and the various components can be represented as

$$T_{\tau_x} = \hat{i} \bullet \frac{D}{Dt} \int_m \mathbf{r} \times \mathbf{U} dm \quad (3.48)$$

In the same way the component in  $y$  and  $z$  can be obtained.

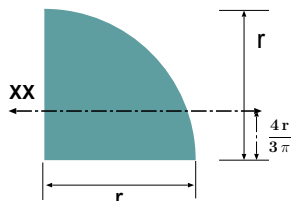
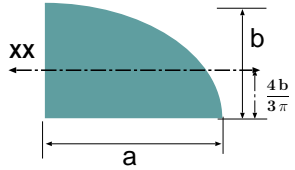
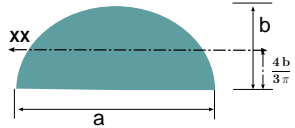
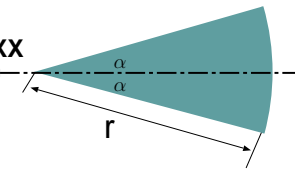
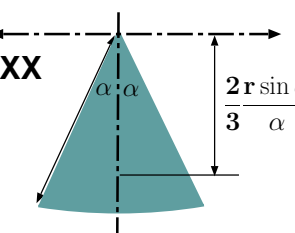
### 3.5.1 Tables of geometries

The following tables present several moment of inertias of commonly used geometries.

Table -3.1. Moments of Inertia for various plane surfaces about their center of gravity (full shapes)

Shape Name	Picture description	$x_c, y_c$	A	$I_{xx}$
Rectangle		$\frac{b}{2}; \frac{a}{2}$	$ab$	$\frac{ab^3}{12}$
Triangle		$\frac{a}{3}$	$\frac{ab}{3}$	$\frac{ab^3}{36}$
Circle		$\frac{b}{2}$	$\frac{\pi b^2}{4}$	$\frac{\pi b^4}{64}$
Ellipse		$\frac{b}{2} \frac{b}{2}$	$\frac{\pi ab}{4}$	$\frac{Ab^2}{64}$
$y = \alpha x^2$ Parabola		$\frac{3\alpha b}{15\alpha-5}$	$\frac{6\alpha-2}{3} \times \left(\frac{b}{\alpha}\right)^{\frac{3}{2}}$	$\frac{\sqrt{b}(20b^3-14b^2)}{35\sqrt{\alpha}}$

Table -3.2. Moment of inertia for various plane surfaces about their center of gravity

Shape Name	Picture description	$x_c, y_c$	A	$I_{xx}$
Quadrant of Circle		$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$	$r^4 \left( \frac{\pi}{16} - \frac{4}{9\pi} \right)$
Ellipsoidal Quadrant		$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$	$ab^3 \left( \frac{\pi}{16} - \frac{4}{9\pi} \right)$
Half of Elliptic		$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$	$ab^3 \left( \frac{\pi}{16} - \frac{4}{9\pi} \right)$
Circular Sector		0	$2\alpha r^2$	$\frac{r^4}{4} \left( \alpha - \frac{1}{2} \sin 2\alpha \right)$
Circular Sector		$\frac{2}{3} \frac{r \sin \alpha}{\alpha}$	$2\alpha r^2$	$I_{x'x'} = \frac{r^4}{4} \left( \alpha + \frac{1}{2} \sin 2\alpha \right)$