

Note:

CHAPTER 7: MOMENTUM

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“Basics of Fluid Mechanics”

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CHAPTER 6

Momentum Conservation for Control Volume

6.1 Momentum Governing Equation

6.1.1 Introduction to Continuous

In the previous chapter, the Reynolds Transport Theorem (RTT) was applied to mass conservation. Mass is a scalar (quantity without magnitude). This chapter deals with momentum conservation which is a vector. The Reynolds Transport Theorem (RTT) is applicable to any quantity and the discussion here will deal with forces that acting on the control volume. Newton's second law for single body is as the following

$$\mathbf{F} = \frac{d(m\mathbf{U})}{dt} \quad (6.1)$$

It can be noticed that bold notation for the velocity is \mathbf{U} (and not U) to represent that the velocity has a direction. For several bodies (n), Newton's law becomes

$$\sum_{i=1}^n \mathbf{F}_i = \sum_{i=1}^n \frac{d(m\mathbf{U})_i}{dt} \quad (6.2)$$

The fluid can be broken into infinitesimal elements which turn the above equation (6.2) into a continuous form of small bodies which results in

$$\sum_{i=1}^n \mathbf{F}_i = \frac{D}{Dt} \int_{sys} \mathbf{U} \overbrace{\rho dV}^{\text{element mass}} \quad (6.3)$$

Note that the notation D/Dt is used and not d/dt to signify that it referred to a derivative of the system. The Reynold's Transport Theorem (RTT) has to be used on the right hand side.

6.1.2 External Forces

First, the terms on the left hand side, or the forces, have to be discussed. The forces, excluding the external forces, are the body forces, and the surface forces as the following

$$\mathbf{F}_{total} = \mathbf{F}_b + \mathbf{F}_s \quad (6.4)$$

In this book (at least in this discussion), the main body force is the gravity. The gravity acts on all the system elements. The total gravity force is

$$\sum \mathbf{F}_b = \int_{sys} \mathbf{g} \overbrace{\rho dV}^{element\ mass} \quad (6.5)$$

which acts through the mass center towards the center of earth. After infinitesimal time the gravity force acting on the system is the same for control volume, hence,

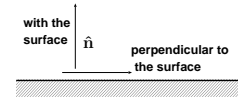
$$\int_{sys} \mathbf{g} \rho dV = \int_{cv} \mathbf{g} \rho dV \quad (6.6)$$

The integral yields a force trough the center mass which has to be found separately.

In this chapter, the surface forces are divided into two categories: one perpendicular to the surface and one with the surface direction (in the surface plain see Figure 6.1.). Thus, it can be written as

$$\sum \mathbf{F}_s = \int_{c.v.} \mathbf{S}_n dA + \int_{c.v.} \boldsymbol{\tau} dA \quad (6.7)$$

Fig. -6.1. The explanation for the direction relative to surface perpendicular and with the surface.



Where the surface "force", \mathbf{S}_n , is in the surface direction, and $\boldsymbol{\tau}$ are the shear stresses. The surface "force", \mathbf{S}_n , is made out of two components, one due to viscosity (solid body) and two consequence of the fluid pressure. Here for simplicity, only the pressure component is used which is reasonable for most situations. Thus,

$$\mathbf{S}_n = -P \hat{n} + \overbrace{\mathbf{S}_\nu}^{\sim 0} \quad (6.8)$$

Where \mathbf{S}_ν is perpendicular stress due to viscosity. Again, \hat{n} is an unit vector outward of element area and the negative sign is applied so that the resulting force acts on the body.

6.1.3 Momentum Governing Equation

The right hand side, according Reynolds Transport Theorem (RTT), is

$$\frac{D}{Dt} \int_{sys} \rho \mathbf{U} dV = \frac{t}{dt} \int_{c.v.} \rho \mathbf{U} dV + \int_{c.v.} \rho \mathbf{U} \mathbf{U}_{rn} dA \quad (6.9)$$

The liquid velocity, \mathbf{U} , is measured in the frame of reference and \mathbf{U}_{rn} is the liquid relative velocity to boundary of the control volume measured in the same frame of reference.

Thus, the general form of the momentum equation without the external forces is

Integral Momentum Equation

$$\int_{c.v.} \mathbf{g} \rho dV - \int_{c.v.} \mathbf{P} dA + \int_{c.v.} \boldsymbol{\tau} \cdot d\mathbf{A} = \frac{t}{dt} \int_{c.v.} \rho \mathbf{U} dV + \int_{c.v.} \rho \mathbf{U} \mathbf{U}_{rn} dV \quad (6.10)$$

With external forces equation (6.10) is transformed to

Integral Momentum Equation & External Forces

$$\sum \mathbf{F}_{ext} + \int_{c.v.} \mathbf{g} \rho dV - \int_{c.v.} \mathbf{P} \cdot d\mathbf{A} + \int_{c.v.} \boldsymbol{\tau} \cdot d\mathbf{A} = \frac{t}{dt} \int_{c.v.} \rho \mathbf{U} dV + \int_{c.v.} \rho \mathbf{U} \mathbf{U}_{rn} dV \quad (6.11)$$

The external forces, F_{ext} , are the forces resulting from support of the control volume by non-fluid elements. These external forces are commonly associated with pipe, ducts, supporting solid structures, friction (non-fluid), etc.

Equation (6.11) is a vector equation which can be broken into its three components. In Cartesian coordinate, for example in the x coordinate, the components are

$$\sum F_x + \int_{c.v.} (\mathbf{g} \cdot \hat{i}) \rho dV - \int_{c.v.} \mathbf{P} \cos \theta_x dA + \int_{c.v.} \tau_x \cdot d\mathbf{A} = \frac{t}{dt} \int_{c.v.} \rho U_x dV + \int_{c.v.} \rho U_x \cdot \mathbf{U}_{rn} dA \quad (6.12)$$

where θ_x is the angle between \hat{n} and \hat{i} or $(\hat{n} \cdot \hat{i})$.

6.1.4 Momentum Equation in Acceleration System

For accelerate system, the right hand side has to include the following acceleration

$$\mathbf{a}_{acc} = \boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) + 2\mathbf{U} \times \boldsymbol{\omega} + \mathbf{r} \times \dot{\boldsymbol{\omega}} - \mathbf{a}_0 \quad (6.13)$$

Where \mathbf{r} is the distance from the center of the frame of reference and the add force is

$$\mathbf{F}_{add} = \int_{V_{c.v.}} \mathbf{a}_{acc} \rho dV \quad (6.14)$$

Integral of Uniform Pressure on Body

In this kind of calculations, it common to obtain a situation where one of the term will be an integral of the pressure over the body surface. This situation is a similar idea that was shown in Section 4.6. In this case the resulting force due to the pressure is zero to all directions.

6.1.5 Momentum For Steady State and Uniform Flow

The momentum equation can be simplified for the steady state condition as it was shown in example 6.3. The unsteady term (where the time derivative) is zero.

Integral Steady State Momentum Equation

$$\sum \mathbf{F}_{ext} + \int_{c.v.} \mathbf{g} \rho dV - \int_{c.v.} \mathbf{P} dA + \int_{c.v.} \boldsymbol{\tau} dA = \int_{c.v.} \rho \mathbf{U} U_{rn} dA \quad (6.15)$$

6.1.5.1 Momentum for For Constant Pressure and Frictionless Flow

Another important sub category of simplification deals with flow under approximation of the frictionless flow and uniform pressure. This kind of situations arise when friction (forces) is small compared to kinetic momentum change. Additionally, in these situations, flow is exposed to the atmosphere and thus (almost) uniform pressure surrounding the control volume. In this situation, the mass flow rate in and out are equal. Thus, equation (6.15) is further reduced to

$$\mathbf{F} = \int_{out} \rho \mathbf{U} \overbrace{(\mathbf{U} \cdot \hat{n})}^{U_{rn}} dA - \int_{in} \rho \mathbf{U} \overbrace{(\mathbf{U} \cdot \hat{n})}^{U_{rn}} dA \quad (6.16)$$

In situations where the velocity is provided and known (remember that density is constant) the integral can be replaced by

$$\mathbf{F} = \dot{m} \overline{\mathbf{U}}_o - \dot{m} \overline{\mathbf{U}}_i \quad (6.17)$$

The average velocity is related to the velocity profile by the following integral

$$\overline{U}^2 = \frac{1}{A} \int_A [U(r)]^2 dA \quad (6.18)$$

Equation (6.18) is applicable to any velocity profile and any geometrical shape.

Example 6.1:

Calculate the average velocity for the given parabolic velocity profile for a circular pipe.

SOLUTION

The velocity profile is

$$U\left(\frac{r}{R}\right) = U_{max} \left[1 - \left(\frac{r}{R}\right)^2\right] \tag{6.1.a}$$

Substituting equation (6.1.a) into equation (6.18)

$$\bar{U}^2 = \frac{1}{2\pi R^2} \int_0^R [U(r)]^2 2\pi r dr \tag{6.1.b}$$

results in

$$\bar{U}^2 = (U_{max})^2 \int_0^1 (1 - \bar{r}^2)^2 \bar{r} d\bar{r} = \frac{1}{6} (U_{max})^2 \tag{6.1.c}$$

Thus,

$$\bar{U} = \frac{U_{max}}{\sqrt{6}}$$

————— End Solution —————

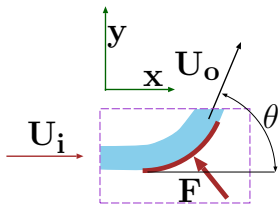


Fig a. Schematics of area impinged by a jet for example 6.2.

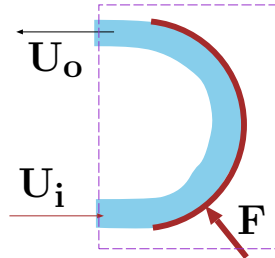


Fig b. Schematics of maximum angle for impinged by a jet.

Fig. -6.2. Schematics of area impinged by a jet and angle effects.

Example 6.2:

A jet is impinging on a stationary surface by changing only the jet direction (see Figure 6.2). Neglect the friction, calculate the force and the angle which the support has to apply to keep the system in equilibrium. What is the angle for which maximum force will be created?

SOLUTION

Equation (6.11) can be reduced, because it is a steady state, to

$$\mathbf{F} = \int_{out} \rho \mathbf{U} \overbrace{(\mathbf{U} \cdot \hat{n})}^{U_{rn}} dA - \int_{in} \rho \mathbf{U} \overbrace{(\mathbf{U} \cdot \hat{n})}^{U_{rn}} dA = \dot{m} \mathbf{U}_o - \dot{m} \mathbf{U}_i \quad (6.11.a)$$

It can be noticed that even though the velocity change direction, the mass flow rate remains constant. Equation (6.11.a) can be explicitly written for the two coordinates. The equation for the x coordinate is

$$F_x = \dot{m} (\cos \theta U_o - U_i)$$

or since $U_i = U_o$

$$F_x = \dot{m} U_i (\cos \theta - 1)$$

It can be observed that the maximum force, F_x occurs when $\cos \theta = \pi$. It can be proven by setting $dF_x/d\theta = 0$ which yields $\theta = 0$ a minimum and the previous solution. Hence

$$F_x|_{max} = -2 \dot{m} U_i$$

and the force in the y direction is

$$F_y = \dot{m} U_i \sin \theta$$

the combined forces are

$$F_{total} = \sqrt{F_x^2 + F_y^2} = \dot{m} U_i \sqrt{(\cos \theta - 1)^2 + \sin^2 \theta}$$

Which results in

$$F_{total} = \dot{m} U_i \sin(\theta/2)$$

with the force angle of

$$\tan \phi = \pi - \frac{F_y}{F_x} = \frac{\pi}{2} - \frac{\theta}{2}$$

For angle between $0 < \theta < \pi$ the maximum occur at $\theta = \pi$ and the minimum at $\theta \sim 0$. For small angle analysis is important in the calculations of flow around thin wings.

End Solution

Example 6.3:

Liquid flows through a symmetrical nozzle as shown in the Figure 6.3 with a mass

6.1. MOMENTUM GOVERNING EQUATION

flow rate of $0.01 [gk/sec]$. The entrance pressure is $3[Bar]$ and the entrance velocity is $5 [m/sec]$. The exit velocity is uniform but unknown. The exit pressure is $1[Bar]$. The entrance area is $0.0005[m^2]$ and the exit area is $0.0001[cm^2]$. What is the exit velocity? What is the force acting the nozzle? Assume that the density is constant $\rho = 1000[kg/m^3]$ and the volume in the nozzle is $0.0015 [m^3]$.

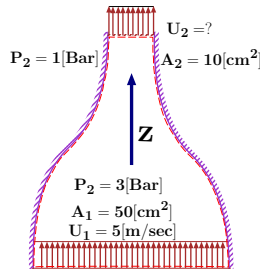


Fig. -6.3. Nozzle schematic for the discussion on the forces and for example 6.3.

SOLUTION

The chosen control volume is shown in Figure 6.3. First, the velocity has to be found. This situation is a steady state for constant density. Then

$$A_1 U_1 = A_2 U_2$$

and after rearrangement, the exit velocity is

$$U_2 = \frac{A_1}{A_2} U_1 = \frac{0.0005}{0.0001} \times 5 = 25[m/sec]$$

Equation (6.12) is applicable but should be transformed into the z direction which is

$$\sum F_z + \int_{c.v.} \mathbf{g} \cdot \hat{k} \rho dV + \int_{c.v.} \mathbf{P} \cos \theta_z dA + \int_{c.v.} \boldsymbol{\tau}_z dA = \underbrace{0}_{=0} + \frac{t}{dt} \int_{c.v.} \rho \mathbf{U}_z dV + \int_{c.v.} \rho \mathbf{U}_z \cdot \mathbf{U}_{rn} dA \tag{6.III.a}$$

The control volume does not cross any solid body (or surface) there is no external forces. Hence,

$$\underbrace{\sum F_z}_{=0} + \int_{c.v.} \mathbf{g} \cdot \hat{k} \rho dV + \underbrace{\int_{c.v.} \mathbf{P} \cos \theta_z dA + \int_{c.v.} \boldsymbol{\tau}_z dA}_{\substack{\text{liquid surface} \\ \text{forces on} \\ \text{the nozzle} \\ F_{nozzle}}} = \underbrace{\int_{c.v.} \mathbf{P} \cos \theta_z dA + \int_{c.v.} \boldsymbol{\tau}_z dA}_{\substack{\text{solid} \\ \text{surface}}} = \int_{c.v.} \rho \mathbf{U}_z \cdot \mathbf{U}_{rn} dA \tag{6.III.b}$$

All the forces that act on the nozzle are combined as

$$\sum F_{nozzle} + \int_{c.v.} \mathbf{g} \cdot \hat{k} \rho dV + \int_{c.v.} \mathbf{P} \cos \theta_z dA = \int_{c.v.} \rho \mathbf{U}_z \cdot \mathbf{U}_{rn} dA \quad (6.III.c)$$

The second term or the body force which acts through the center of the nozzle is

$$\mathbf{F}_b = - \int_{c.v.} \mathbf{g} \cdot \hat{n} \rho dV = -g \rho V_{nozzle}$$

Notice that in the results the gravity is not bold since only the magnitude is used. The part of the pressure which act on the nozzle in the z direction is

$$- \int_{c.v.} P dA = \int_1 P dA - \int_2 P dA = PA|_1 - PA|_2$$

The last term in equation (6.III.c) is

$$\int_{c.v.} \rho \mathbf{U}_z \cdot \mathbf{U}_{rn} dA = \int_{A_2} U_2 (U_2) dA - \int_{A_1} U_1 (U_1) dA$$

which results in

$$\int_{c.v.} \rho \mathbf{U}_z \cdot \mathbf{U}_{rn} dA = \rho (U_2^2 A_2 - U_1^2 A_1)$$

Combining all transform equation (6.III.c) into

$$F_z = -g \rho V_{nozzle} + PA|_2 - PA|_1 + \rho (U_2^2 A_2 - U_1^2 A_1) \quad (6.III.d)$$

$$F_z = 9.8 \times 1000 \times$$

End Solution

6.2 Momentum Equation Application

Momentum Equation Applied to Propellers

The propeller is a mechanical devise that is used to increase the fluid momentum. Many times it is used for propulsion purposes of airplanes, ships and other devices (thrust) as shown in Figure 6.4. The propeller can be stationary like in cooling tours, fan etc. The other common used of propeller is mostly to move fluids as a pump.

The propeller analysis of unsteady is complicated due to the difficulty in understanding the velocity field. For a steady state the analysis is simpler and used here to provide an example of steady state. In the Figure 6.4 the fluid flows from the left to the right. Either it is assumed that some of the fluid enters into the container and fluid outside is not affected by the propeller. Or there is a line (or surface) in which the fluid outside changes only the flow direction. This surface is called slip surface. Of course it is only approximation but is provided a crude tool. Improvements can be made to this analysis. Here, this analysis is used for academic purposes.

As first approximation, the pressure around control volume is the same. Thus, pressure drops from the calculation. The one dimensional momentum equation is reduced

$$F = \rho (U_2^2 - U_1^2) \quad (6.19)$$

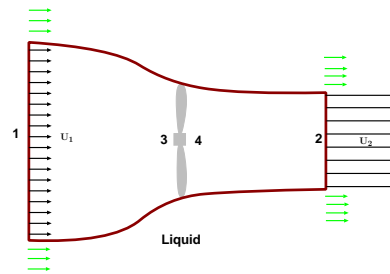


Fig. -6.4. Propeller schematic to explain the change of momentum due to velocity.

Combining the control volume between points 1 and 3 with (note that there are no external forces) with points 4 and 2 results in

$$\rho (U_2^2 - U_1^2) = P_4 - P_3 \quad (6.20)$$

This analysis provide way to calculate the work needed to move this propeller. Note that in this analysis it was assumed that the flow is horizontal that $z_1 = z_2$ and/or the change is insignificant.

Jet Propulsion

Jet propulsion is a mechanism in which the air planes and other devices are propelled. Essentially, the air is sucked into engine and with addition heating (burning fuel) the velocity is increased. Further increase of the exit area with the increased of the burned gases further increase the thrust. The analysis of such device in complicated and there is a whole class dedicated for such topic in many universities. Here, a very limited discussion related to the steady state is offered.

The difference between the jets propulsion and propellers is based on the energy supplied. The propellers are moved by a mechanical work which is converted to thrust. In Jet propulsion, the thermal energy is converted to thrust. Hence, this direct conversion can be, and is, in many case more efficient. Furthermore, as it will be shown in the Chapter on compressible flow it allows to achieve velocity above speed of sound, a major obstacle in the past.

The inlet area and exit area are different for most jets and if the mass of the fuel is neglected then

$$F = \rho (A_2 U_2^2 - A_1 U_1^2) \quad (6.21)$$

An academic example to demonstrate how a steady state calculations are done for a moving control volume. Notice that

Example 6.4:

A sled toy shown in Figure 6.5 is pushed by liquid jet. Calculate the friction force on the

toy when the toy is at steady state with velocity, U_0 . Assume that the jet is horizontal and the reflecting jet is vertical. The velocity of the jet is uniform. Neglect the friction between the liquid (jet) and the toy and between the air and toy. Calculate the absolute velocity of the jet exit. Assume that the friction between the toy and surface (ground) is relative to the vertical force. The dynamics friction is μ_d .

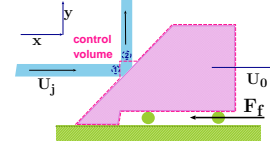


Fig. -6.5. Toy Sled pushed by the liquid jet in a steady state for example 6.4.

SOLUTION

The chosen control volume is attached to the toy and thus steady state is obtained. The frame of reference is moving with the toy velocity, U_0 . The applicable mass conservation equation for steady state is

$$A_1 U_1 = A_2 U_2$$

The momentum equation in the x direction is

$$\mathbf{F}_f + \int_{c.v.} \mathbf{g} \rho dV - \int_{c.v.} \mathbf{P} dA + \int_{c.v.} \boldsymbol{\tau} dA = \int_{c.v.} \rho \mathbf{U} \mathbf{U}_{rn} dV \quad (6.IV.a)$$

The relative velocity into the control volume is

$$\mathbf{U}_{1j} = (U_j - U_0) \hat{x}$$

The relative velocity out the control volume is

$$\mathbf{U}_{2j} = (U_j - U_0) \hat{y}$$

The absolute exit velocity is

$$\mathbf{U}_2 = U_0 \hat{x} + (U_j - U_0) \hat{y}$$

For small volume, the gravity can be neglected also because this term is small compared to other terms, thus

$$\int_{c.v.} \mathbf{g} \rho dV \sim 0$$

The same can be said for air friction as

$$\int_{c.v.} \boldsymbol{\tau} dA \sim 0$$

The pressure is uniform around the control volume and thus the integral is

$$\int_{c.v.} \mathbf{P} dA = 0$$

The control volume was chosen so that the pressure calculation is minimized.

The momentum flux is

$$\int_{S_{c.v.}} \rho U_x U_i r n dA = A \rho U_{1j}^2 \quad (6.IV.b)$$

The substituting (6.IV.b) into equation (6.IV.a) yields

$$F_f = A \rho U_{1j}^2 \quad (6.IV.c)$$

The friction can be obtained from the momentum equation in the y direction

$$m_{toy} g + A \rho U_{1j}^2 = F_{earth}$$

According to the statement of question the friction force is

$$F_f = \mu_d (m_{toy} g + A \rho U_{1j}^2)$$

The momentum in the x direction becomes

$$\mu_d (m_{toy} g + A \rho U_{1j}^2) = A \rho U_{1j}^2 = A \rho (U_j - U_0)^2$$

The toy velocity is then

$$U_0 = U_j - \sqrt{\frac{\mu_d m_{toy} g}{A \rho (1 - \mu_d)}}$$

Increase of the friction reduce the velocity. Additionally larger toy mass decrease the velocity.

End Solution

6.2.1 Momentum for Unsteady State and Uniform Flow

The main problem in solving the unsteady state situation is that the control volume is accelerating. A possible way to solve the problem is by expressing the terms in an equation (6.10). This method is cumbersome in many cases. Alternative method of solution is done by attaching the frame of reference to the accelerating body. One such example of such idea is associated with the Rocket Mechanics which is present here.

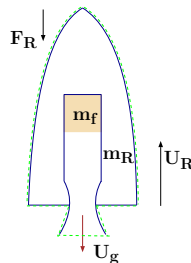


Fig. -6.6. A rocket with a moving control volume.

6.2.2 Momentum Application to Unsteady State

Rocket Mechanics

A rocket is a device similar to jet propulsion. The difference is the fact that the oxidant is on board with the fuel. The two components are burned and the gases are ejected through a nozzle. This mechanism is useful for specific locations because it is independent of the medium through which it travels. In contrast to other mechanisms such as jet propulsion which obtain the oxygen from the medium which they travel the rockets carry the oxygen with it. The rocket is accelerating and thus the frame for reference is moving with the rocket. The velocity of the rocket in the rocket frame of reference is zero. However, the derivative with respect to time, $d\mathbf{U}/dt \neq 0$ is not zero. The resistance of the medium is Denote as F_R . The momentum equation is

$$\underbrace{\int_{c.v.} \tau dA}_{F_R} + \int_{c.v.} \mathbf{g} \rho dV + \underbrace{\int_{c.v.} \mathbf{P} dA}_0 - \int \rho a_0 dV = \frac{d}{dt} \int_{V_{c.v.}} \rho U_y dV + \int_{c.v.} \rho U_y U_{rn} dA \quad (6.22)$$

There are no external forces in this control volume thus, the first term F_R , vanishes. The pressure term vanishes because the pressure essentially is the same and the difference can be neglected. The gravity term is an instantaneous mass times the gravity times the constant and the same can be said for the acceleration term. Yet, the acceleration is the derivative of the velocity and thus

$$\int \rho a_0 dV = \frac{dU}{dt} (m_R + m_f) \quad (6.23)$$

The first term on the right hand side is the change of the momentum in the rocket volume. This change is due to the change in the volume of the oxidant and the fuel.

$$\frac{d}{dt} \int_{V_{c.v.}} \rho U_y dV = \frac{d}{dt} [(m_R + m_f) U] \quad (6.24)$$

Clearly, the change of the rocket mass can be considered minimal or even neglected. The oxidant and fuel flow outside. However, inside the rocket the change in the velocity is due to change in the reduction of the volume of the oxidant and fuel. This change is minimal and for this analysis, it can be neglected. The last term is

$$\int_{c.v.} \rho U_y U_{rn} dA = \dot{m} (U_g - U_R) \quad (6.25)$$

Combining all the above terms results in

$$-F_R - (m_R + m_f) g + \frac{dU}{dt} (m_R + m_f) = \dot{m} (U_g - U_R) \quad (6.26)$$

Denoting $\mathcal{M}_T = m_R + m_f$ and thus $d\mathcal{M}/dt = \dot{m}$ and $U_e = U_g - U_R$. As first approximation, for constant fuel consumption (and almost oxidant), gas flow out is constant as well. Thus, for constant constant gas consumption equation (6.26) transformed to

$$-F_R - \mathcal{M}_T g + \frac{dU}{dt} \mathcal{M}_T = \dot{\mathcal{M}}_T U_e \quad (6.27)$$

Separating the variables equation (6.27) yields

$$dU = \left(\frac{-\dot{\mathcal{M}}_T U_e}{\mathcal{M}_T} - \frac{F_R}{\mathcal{M}_T} - g \right) dt \quad (6.28)$$

Before integrating equation (6.28), it can be noticed that the friction resistance F_R , is a function of the several parameters such the duration, the speed (the Reynolds number), material that surface made and the medium it flow in altitude. For simplicity here the part close to Earth (to the atmosphere) is assumed to be small compared to the distance in space. Thus it is assume that $F_R = 0$. Integrating equation (6.28) with limits of $U(t = 0) = 0$ provides

$$\int_0^U dU = -\dot{\mathcal{M}}_T U_e \int_0^t \frac{dt}{\mathcal{M}_T} - \int_0^t g dt \quad (6.29)$$

the results of the integration is (notice $\mathcal{M} = \mathcal{M}_0 - t \dot{\mathcal{M}}$)

$$U = U_e \ln \left(\frac{\mathcal{M}_0}{\mathcal{M}_0 - t \dot{\mathcal{M}}} \right) - g t \quad (6.30)$$

The following is an elaborated example which deals with an unsteady two dimensional problem. This problem demonstrates the used of control volume to find method of approximation for not given velocity profiles¹

Example 6.5:

¹A variation of this problem has appeared in many books in the literature. However, in the past it was not noticed that a slight change in configuration leads to a constant x velocity. This problem was aroused in manufacturing industry. This author was called for consultation and to solve a related problem. For which he noticed this "constant velocity."

A tank with wheels is filled with liquid is depicted in Figure 6.7. The tank upper part is opened to the atmosphere. At initial time the valve on the tank is opened and the liquid flows out with an uniform velocity profile. The tank mass with the wheels (the solid parts) is known, m_t . Calculate the tank velocity for two cases. One the wheels have a constant resistance with the ground and two the resistance linear function of the weight. Assume that the exit velocity is a linear function of the height.

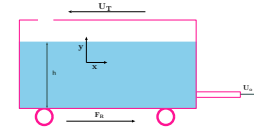


Fig. -6.7. Schematic of a tank seating on wheel for unsteady state discussion

SOLUTION

This problem is similar to the rocket mechanics with a twist, the source of the propulsion is the potential energy. Furthermore, the fluid has two velocity components verse one component in the rocket mechanics. The control volume is shown in Figure 6.7. The frame of reference is moving with the tank. This situation is unsteady state thus equation (6.12) for two dimensions is used. The mass conservation equation is

$$\frac{d}{dt} \int_{V_{c.v.}} \rho dV + \int_{S_{c.v.}} \rho dA = 0 \quad (6.V.a)$$

Equation (6.V.a) can be transferred to

$$\frac{dm_{c.v.}}{dt} = -\rho U_0 A_0 = -m_0 \quad (6.V.b)$$

Where m_0 is mass flow rate out. Equation (6.V.b) can be further reduced due to constant density to

$$\frac{d(Ah)}{dt} + U_0 A_0 = 0 \quad (6.V.c)$$

It can be noticed that the area of the tank is almost constant ($A = constant$) thus

$$A \frac{dh}{dt} + U_0 A_0 = 0 \implies \frac{dh}{dt} = -\frac{U_0 A_0}{A} \quad (6.31)$$

The relationship between the height and the flow now can be used.

$$U_0 = \mathcal{B} h \quad (6.V.d)$$

Where \mathcal{B} is the coefficient that has the right units to mach equation (6.V.d) that represent the resistance in the system and substitute the energy equation. Substituting equation (6.V.d) into equation (6.V.c) results in

$$\frac{dh}{dt} + \frac{\mathcal{B} h A_0}{A} = 0 \quad (6.V.e)$$

Equation (6.V.e) is a first order differential equation which can be solved with the initial condition $h(t = 0) = h_0$. The solution (see for details in the Appendix A.2.1) is

$$h(t) = h_0 e^{-\frac{t A_0 \mathcal{B}}{A}} \tag{6.V.f}$$

To find the average velocity in the x direction a new control volume is used. The boundary of this control volume are the tank boundary on the left with the straight surface as depicted in Figure 6.8. The last boundary is variable surface in a distance x from the tank left part. The tank depth, is not relevant. The mass conservation for this control volume is

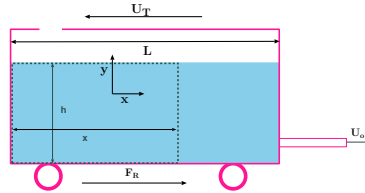


Fig. -6.8. A new control volume to find the velocity in discharge tank for example 6.5.

$$w x \frac{dh}{dt} = -w h \bar{U}_x \tag{6.V.g}$$

Where here w is the depth or width of the tank. Substituting (6.V.f) into (6.V.g) results

$$\bar{U}_x(x) = \frac{x A_0 h_0 \mathcal{B}}{A h} e^{-\frac{t A_0 \mathcal{B}}{A}} = \frac{x A_0 \mathcal{B}}{A} \tag{6.V.h}$$

The average x component of the velocity is a linear function of x . Perhaps surprising, it also can be noticed that $\bar{U}_x(x)$ is a not function of the time. Using this function, the average velocity in the tank is

$$\bar{U}_x = \frac{1}{L} \int_0^L \frac{x A_0 \mathcal{B}}{A} = \frac{L A_0 \mathcal{B}}{2 A} \tag{6.V.i}$$

It can be noticed that \bar{U}_x is not function of height, h . In fact, it can be shown that average velocity is a function of cross section (what direction?).

Using a similar control volume², the average velocity in the y direction is

$$\bar{U}_y = \frac{dh}{dt} = -\frac{h_0 A_0 \mathcal{B}}{A} e^{-\frac{t A_0 \mathcal{B}}{A}} \tag{6.V.j}$$

It can be noticed that the velocity in the y is a function of time as oppose to the x direction.

The applicable momentum equation (in the tank frame of reference) is (6.11) which is reduced to

$$-\mathbf{F}_R - (m_t + m_f) \mathbf{g} - \overbrace{\mathbf{a} (m_t + m_f)}^{\text{acceleration}} = \frac{d}{dt} [(m_t + m_f) \mathbf{U}_r] + U_o m_o \tag{6.V.k}$$

²The boundaries are the upper (free surface) and tank side with a y distance from the free surface. $\int U_{bn} dA = \int U_{rn} dA \implies U_{bn} = U_{rn}$.

Where \mathbf{U}_r is the relative fluid velocity to the tank (if there was no tank movement). m_f and m_t are the mass of the fluid and the mass of tank respectively. The acceleration of the tank is $\mathbf{a} = -\hat{i}a_0$ or $\hat{i} \cdot \mathbf{a} = -a$. And the additional force for accelerated system is

$$-\hat{i} \cdot \int_{V_{c.v.}} \mathbf{a} \rho dV = m_{c.v.} a$$

The mass in the control volume include the mass of the liquid with mass of the solid part (including the wheels).

$$m_{c.v.} = m_f + m_T$$

because the density of the air is very small the change of the air mass is very small as well ($\rho_a \ll \rho$).

The pressure around the control volume is uniform thus

$$\int_{S_{c.v.}} P \cos \theta_x dA \sim 0$$

and the resistance due to air is negligible, hence

$$\int_{S_{c.v.}} \tau dA \sim 0$$

The momentum flow rate out of the tank is

$$\int_{S_{c.v.}} \rho U_x U_{rn} dA = \rho U_o^2 A_o = m_o U_o \quad (6.32)$$

In the x coordinate the momentum equation is

$$-F_x + (m_t + m_f) a = \frac{d}{dt} [(m_t + m_f) U_x] + U_o \dot{m}_f \quad (6.V.1)$$

Where F_x is the x component of the reaction which is opposite to the movement direction. The momentum equation in the y coordinate it is

$$F_y - (m_t + m_f) g = \frac{d}{dt} [(m_t + m_f) U_y] \quad (6.V.m)$$

There is no mass flow in the y direction and U_y is component of the velocity in the y direction.

The tank movement cause movement of the air which cause momentum change. This momentum is function of the tank volume times the air density times tank velocity ($h_0 \times A \times \rho_a \times U$). This effect is known as the add mass/momentum and will be discussed in the Dimensional Analysis and Ideal Flow Chapters. Here this effect is neglected.

The main problem of integral analysis approach is that it does not provide a way to analysis the time derivative since the velocity profile is not given inside the control volume. This limitation can be partially overcome by assuming some kind of average. It

can be noticed that the velocity in the tank has two components. The first component is downward (y) direction and the second in the exit direction (x). The velocity in the y direction does not contribute to the momentum in the x direction. The average velocity in the tank (because constant density and more about it later section) is

$$\bar{U}_x = \frac{1}{V_t} \int_{V_f} U_x dV$$

Because the integral is replaced by the average it is transferred to

$$\int_{V_f} \rho U_x dV \sim m_{c.v.} \bar{U}_x$$

Thus, if the difference between the actual and averaged momentum is neglected then

$$\frac{d}{dt} \int_{V_f} \rho U_x dV \sim \frac{d}{dt} (m_{c.v.} \bar{U}_x) = \frac{d m_{c.v.}}{dt} \bar{U}_x + \overbrace{\frac{d \bar{U}_x}{dt}}^{\sim 0} m_{c.v.} \quad (6.V.n)$$

Noticing that the derivative with time of control volume mass is the flow out in equation (6.V.n) becomes

$$\frac{d m_{c.v.}}{dt} \bar{U}_x + \frac{d \bar{U}_x}{dt} m_{c.v.} = - \overbrace{\dot{m}_0}^{\text{mass rate out}} \bar{U}_x = -m_0 \frac{L A_0 \mathcal{B}}{2 A} \quad (6.V.o)$$

Combining all the terms results in

$$-F_x + a (m_f + m_t) = -m_0 \frac{L A_0 \mathcal{B}}{2 A} - U_0 m_0 \quad (6.V.p)$$

Rearranging and noticing that $a = dU_T/dt$ transformed equation (6.V.p) into

$$a = \frac{F_x}{m_f + m_t} - m_0 \left(\frac{L A_0 \mathcal{B} + 2 A U_0 (m_f + m_t)}{2 A (m_f + m_t)} \right) \quad (6.V.q)$$

If the $F_x \geq m_0 \left(\frac{L A_0 \mathcal{B}}{2 A} + U_0 \right)$ the toy will not move. However, if it is the opposite the toy start to move. From equation (6.V.d) the mass flow out is

$$m_0(t) = \mathcal{B} h_0 \overbrace{e^{-\frac{U_0}{h} t}}^{\frac{U_0}{h}} \overbrace{\frac{A_0 \mathcal{B}}{A}}^{\frac{h}{t A_0 \mathcal{B}}} A_0 \rho \quad (6.V.r)$$

The mass in the control volume is

$$m_f = \rho A h_0 \overbrace{e^{-\frac{U_0}{h} t}}^{\frac{U_0}{h}} \overbrace{\frac{A_0 \mathcal{B}}{A}}^{\frac{h}{t A_0 \mathcal{B}}} \quad (6.V.s)$$

The initial condition is that $U_T(t = 0) = 0$. Substituting equations (6.V.r) and (6.V.s) into equation (6.V.q) transforms it to a differential equation which is integrated if R_x is constant.

For the second case where R_x is a function of the R_y as

$$R_x = \mu R_y \quad (6.33)$$

The y component of the average velocity is function of the time. The change in the accumulative momentum is

$$\frac{d}{dt} [(m_f) \bar{U}_y] = \frac{dm_f}{dt} \bar{U}_y + \frac{d\bar{U}_y}{dt} m_f \quad (6.V.t)$$

The reason that m_f is used because the solid parts do not have velocity in the y direction. Rearranging the momentum equation in the y direction transformed

$$F_y = \left(m_t + \rho A h_0 \overbrace{e^{-\frac{t A_0 \mathcal{B}}{A}}}^{m_f} \right) g + 2 \left(\frac{\rho h_0 A_0^2 \mathcal{B}^2}{A} \right)^2 e^{-\frac{t A_0 \mathcal{B}}{A}} \quad (6.V.u)$$

The actual results of the integrations are not provided since the main purpose of this exercise is to learn how to use the integral analysis.

End Solution

Averaged Velocity! Estimates

In example 6.1 relationship between momentum of maximum velocity to average velocity was presented. Here, relationship between momentum for the average velocity to the actual velocity is presented. There are situations where actual velocity profile is not known but is function can be approximated. For example, the velocity profile can be estimated using the ideal fluid theory but the actual values are not known. For example, the flow profile in example 6.5 can be estimated even by hand sketching.

For these cases a correction factor can be used. This correction factor can be calculated by finding the relation between the two cases. The momentum for average velocity is

$$M_a = m_{c.v} \bar{U} = \rho V \int_{c.v} U dV \quad (6.34)$$

The actual momentum for control volume is

$$M_c = \int_{c.v.} \rho U_x dV \quad (6.35)$$

These two have to equal thus,

$$\mathcal{C} \rho V \int_{c.v} U dV = \int_{c.v.} \rho U_x dV \quad (6.36)$$

If the density is constant then the coefficient is one ($\mathcal{C} \equiv 1$). However, if the density is not constant, the coefficient is not equal to one.

6.3 Conservation Moment Of Momentum

The angular momentum can be derived in the same manner as the momentum equation for control volume. The force

$$\mathbf{F} = \frac{D}{Dt} \int_{V_{sys}} \rho \mathbf{U} dV \quad (6.37)$$

The angular momentum then will be obtained by calculating the change of every element in the system as

$$\mathfrak{M} = \mathbf{r} \times \mathbf{F} = \frac{D}{Dt} \int_{V_{sys}} \rho \mathbf{r} \times \mathbf{U} dV \quad (6.38)$$

Now the left hand side has to be transformed into the control volume as

$$\mathfrak{M} = \frac{d}{dt} \int_{V_{c.v.}} \rho (\mathbf{r} \times \mathbf{U}) dV + \int_{S_{c.v.}} \rho (\mathbf{r} \times \mathbf{U}) \mathbf{U}_{rn} dA \quad (6.39)$$

The angular momentum equation, applying equation (6.39) to uniform and steady state flow with neglected pressure gradient is reduced to

$$\mathfrak{M} = \dot{m} (r_2 \times U_2 + r_1 \times U_1) \quad (6.40)$$

Introduction to Turbo Machinery

The analysis of many turbomachinery such as centrifugal pump is fundamentally based on the angular momentum. To demonstrate this idea, the following discussion is provided. A pump impeller is shown in Figure 6.9 commonly used in industry. The impeller increases the velocity of the fluid by increasing the radius of the particles. The inside particle is obtained larger velocity and due to centrifugal forces is moving to outer radius for which additionally increase of velocity occur. The pressure on the outer side is uniform thus does not create a moment. The flow is assumed to enter the impeller radially with average velocity U_1 . Here it is assumed that fluid is incompressible ($\rho = \text{constant}$). The height of the impeller is h . The exit liquid velocity, U_2 has two components, one the tangential velocity, U_{t2} and radial component, U_{n2} . The relative exit velocity is U_{lr2} and the velocity of the impeller edge is U_{m2} . Notice that tangential liquid velocity, U_{t2} is not equal to the impeller outer edge velocity U_{m2} . It is assumed that required torque is function U_2 , r , and h .

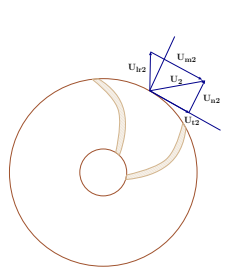


Fig. -6.9. The impeller of the centrifugal pump and the velocities diagram at the exit.

$$\mathfrak{M} = \dot{m} r_2 U_{t2} \quad (6.41)$$

Multiplying equation (6.41) results in

$$\mathfrak{M}\omega = \dot{m} \overbrace{r_2 \omega}^{U_{m2}} U_{t2} \quad (6.42)$$

The shaft work is given by the left side and hence,

$$\dot{W} = \dot{m} U_{m2} U_{t2} \quad (6.43)$$

The difference between U_{m2} to U_{t2} is related to the efficiency of the pump which will be discussed in the chapter on the turbomachinery.

Example 6.6:

A centrifugal pump is pumping $600 \text{ 2}[m^3/\text{hour}]$. The thickness of the impeller, h is $2[\text{cm}]$ and the exit diameter is $0.40[\text{m}]$. The angular velocity is 1200 r.p.m. Assume that angle velocity is leaving the impeller is 125° . Estimate what is the minimum energy required by the pump.

6.4 More Examples on Momentum Conservation

Example 6.7:

A design of a rocket is based on the idea that density increase of the leaving jet increases the acceleration of the rocket see Figure 6.10. Assume that this idea has a good engineering logic. Liquid fills the lower part of the rocket tank. The upper part of the rocket tank is filled with compressed gas. Select the control volume in such a way that provides the ability to find the rocket acceleration. What is the instantaneous velocity of the rocket at time zero? Develop the expression for the pressure (assuming no friction with the walls). Develop expression for rocket velocity. Assume that the gas is obeying the perfect gas model. What are the parameters that effect the problem.

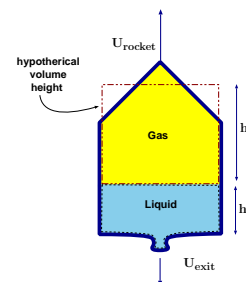


Fig. -6.10. Nozzle schematics water rocket for the discussion on the forces for example 6.7

SOLUTION

Under construction for time being only hints³

In the solution of this problem several assumptions must be made so that the integral system can be employed.

³This problem appeared in the previous version (0.2.3) without a solution. Several people ask to provide a solution or some hints for the solution. The following is not the solution but rather the approach how to treat this problem.

- The surface remained straight at the times and no liquid residue remains behind.
- The gas obeys the ideal gas law.
- The process is isothermal (can be isentropic process).
- No gas leaves the rocket.
- The mixing between the liquid and gas is negligible.
- The gas mass is negligible in comparison to the liquid mass and/or the rocket.
- No resistance to the rocket (can be added).
- The cross section of the liquid is constant.

In this problem the energy source is the pressure of the gas which propels the rocket. Once the gas pressure reduced to be equal or below the outside pressure the rocket have no power for propulsion. Additionally, the initial take off is requires a larger pressure.

The mass conservation is similar to the rocket hence it is

$$\frac{dm}{dt} = -U_e A_e \quad (6.VII.a)$$

The mass conservation on the gas zone is a byproduct of the mass conservation of the liquid. Furthermore, it can be observed that the gas pressure is a direct function of the mass flow out.

The gas pressure at the initial point is

$$P_0 = \rho_0 R T \quad (6.VII.b)$$

Per the assumption the gas mass remain constant and is denoted as m_g . Using the above definition, equation (6.VII.b) becomes

$$P_0 = \frac{m_g R T}{V_{0g}} \quad (6.VII.c)$$

The relationship between the gas volume

$$V_g = \bar{h}_g A \quad (6.VII.d)$$

The gas geometry is replaced by a virtual constant cross section which cross section of the liquid (probably the same as the base of the gas phase). The change of the gas volume is

$$\frac{dV_g}{dt} = A \frac{dh_g}{dt} = -A \frac{dh_\ell}{dt} \quad (6.VII.e)$$

The last identify in the above equation is based on the idea what ever height concede by the liquid is taken by the gas. The minus sign is to account for change of "direction"

of the liquid height. The total change of the gas volume can be obtained by integration as

$$V_g = A (h_{g0} - \Delta h_\ell) \quad (6.VII.f)$$

It must be point out that integral is not function of time since the height as function of time is known at this stage.

The initial pressure now can be expressed as

$$P_0 = \frac{m_g R T}{h_{g0} A} \quad (6.VII.g)$$

The pressure at any time is

$$P = \frac{m_g R T}{h_g A} \quad (6.VII.h)$$

Thus the pressure ratio is

$$\frac{P}{P_0} = \frac{h_{g0}}{h_g} = \frac{h_{g0}}{h_{g0} - \Delta h_\ell} = h_{g0} \frac{1}{1 - \frac{\Delta h_\ell}{h_{g0}}} \quad (6.VII.i)$$

Equation (6.VII.a) can be written as

$$m_\ell(t) = m_{\ell 0} - \int_0^t U_e A_e dt \quad (6.VII.j)$$

From equation (6.VII.a) it also can be written that

$$\frac{dh_\ell}{dt} = \frac{U_e A_e}{\rho_e A} \quad (6.VII.k)$$

According to the assumption the flow out is linear function of the pressure inside thus,

$$U_e = f(P) + g h_\ell \rho \simeq f(P) = \zeta P \quad (6.VII.l)$$

Where ζ here is a constant which the right units.

The liquid momentum balance is

$$-g(m_R + m_\ell) - a(m_R + m_\ell) = \overbrace{\frac{d}{dt}(m_R + m_\ell)U}^{=0} + bc + (U_R + U_\ell) m_\ell \quad (6.VII.m)$$

Where bc is the change of the liquid mass due the boundary movement.

End Solution

Example 6.8:

A rocket is filled with only compressed gas. At a specific moment the valve is opened and the rocket is allowed to fly. What is the minimum pressure which make the rocket fly. What are the parameters that effect the rocket velocity. Develop an expression for the rocket velocity.

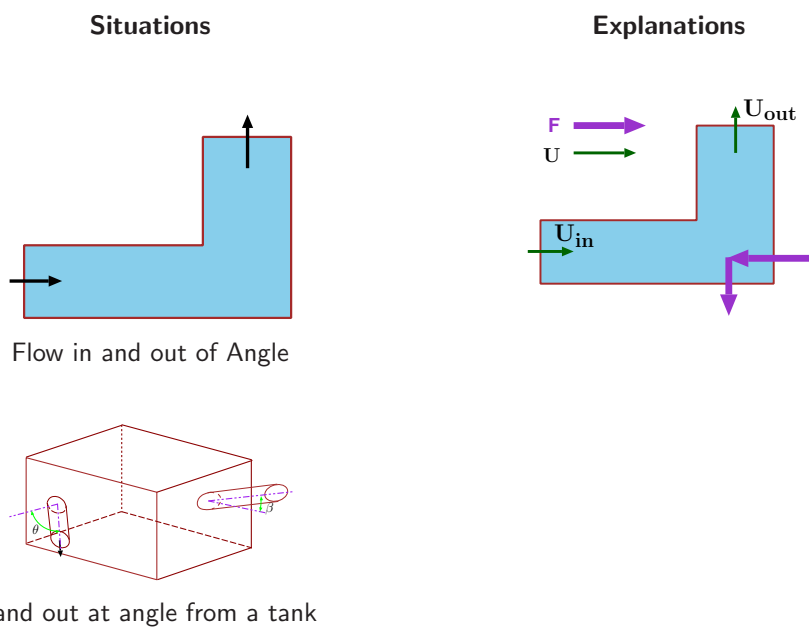
Example 6.9:

In Example 6.5 it was mentioned that there are only two velocity components. What was the assumption that the third velocity component was neglected.

6.4.1 Qualitative Questions

Example 6.10:

For each following figures discuss and state force direction and the momentum that act on the control volume due to .



Example 6.11:

A similar tank as shown in Figure 6.11 is built with a exit located in uneven distance from the the right and the left and is filled with liquid. The exit is located on the left hand side at the front. What are the direction of the forces that keep the control volume in the same location? Hints, consider the unsteady effects. Look at the directions which the unsteady state momentum in the tank change its value.

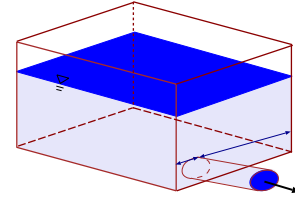


Fig. -6.11. Flow out of un symmetrical tank for example 6.11

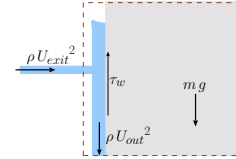
Example 6.12:

A large tank has opening with area, A . In front and against the opening there a block with mass of $50[kg]$. The friction factor between the block and surface is 0.5 . Assume that resistance between the air and the water jet is negligible. Calculated the minimum height of the liquid in the tank in order to start to have the block moving?

SOLUTION

The solution of this kind problem first requires to know at what accuracy this solution is needed. For great accuracy, the effect minor loss or the loss in the tank opening have taken into account. First assuming that a minimum accuracy therefore the information was given on the tank that it large. First, the velocity to move the block can be obtained from the analysis of the block free body diagram (the impinging jet diagram).

The control volume is attached to the block. It is assumed that the two streams in the vertical cancel each other. The jet stream has only one component in the horizontal component. Hence,



$$F = \rho A U_{exit}^2 \tag{6.XII.a}$$

The minimum force to push the block is

Fig. -6.12. Jet impinging jet surface perpendicular and with the surface.

$$\rho A U_{exit}^2 = m g \mu \implies U_{exit} = \sqrt{\frac{m g \mu}{\rho A}} \tag{6.XII.b}$$

And the velocity as a function of the height is $U = \sqrt{\rho g h}$ and thus

$$h = \frac{m \mu}{\rho^2 A} \tag{6.XII.c}$$

It is interesting to point out that the gravity is relevant. That is the gravity has no effect on the velocity (height) required to move the block. However, if the gravity was in the opposite direction, no matter what the height will be the block will not move (neglecting other minor effects). So, the gravity has effect and the effect is the direction, that is the same height will be required on the moon as the earth.

For very tall blocks, the forces that acts on the block in the vertical direction is can be obtained from the analysis of the control volume shown in Figure 6.12. The jet impinged on the surface results in out flow stream going to all the directions in the block surface. Yet, the gravity acts on all these "streams" and eventually the liquid flows downwards. In fact because the gravity the jet impeging in downwards slend direction. At the extreme case, all liquid flows downwards. The balance on the stream downwards (for steady state) is

$$\rho \overline{U_{out}}^2 \cong \rho V_{liquid} g + m g \quad (6.XII.d)$$

Where V_{liquid} is the liquid volume in the control volume (attached to the block). The pressure is canceled because the flow is exposed to air. In cases were $\rho V_{liquid} g > \rho \overline{U_{out}}^2$ the required height is larger. In the oposite cases the height is smaller.

End Solution

