

# Note:

CHAPTER 4: ISENTROPIC

Version: 0.4.8.6    October 23, 2009

This chapter is part of the textbook:

**“Fundamentals of Compressible  
Flow Mechanics”**

**You can download the whole book if you like**

**from:** *www.potto.org*.

*This chapter is under GDL with a minor modifications. Potto License is no longer applied*

*You should be aware that this book is updated about every a few weeks or so.*

GENICK BAR-MEIR, PH.D.  
MINNEAPOLIS, MINNESOTA  
OCTOBER 23, 2009

## THE LIST OF THE AVAILABLE BOOKS IN POTTO PROJECT

Project Name	Progress	Remarks	Version	Availability for Public Download	Number Downloads
Compressible Flow	beta		0.4.8.4	✓	120,000
Die Casting	alpha		0.1	✓	60,000
Dynamics	NSY		0.0.0	✗	-
Fluid Mechanics	alpha		0.1.8	✓	15,000
Heat Transfer	NSY	Based on Eckert	0.0.0	✗	-
Mechanics	NSY		0.0.0	✗	-
Open Channel Flow	NSY		0.0.0	✗	-
Statics	early alpha	first chapter	0.0.1	✗	-
Strength of Material	NSY		0.0.0	✗	-
Thermodynamics	early alpha		0.0.01	✗	-
Two/Multi phases flow	NSY	Tel-Aviv' notes	0.0.0	✗	-

NSY = Not Started Yet

---

---

# CHAPTER 5

---

## Isentropic Flow

In this chapter a discussion on a steady state flow through a smooth and continuous area flow rate is presented. A discussion about the flow through a converging–diverging nozzle is also part of this chapter. The isentropic flow models are important because of two main reasons: One, it provides the information about the trends and important parameters. Two, the correction factors can be introduced later to account for deviations from the ideal state.

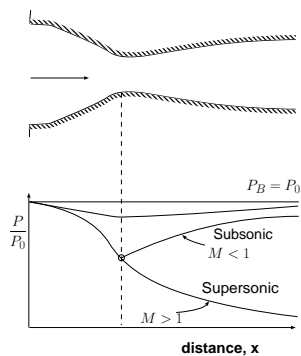


Fig. -5.1. Flow of a compressible substance (gas) through a converging–diverging nozzle.

### 5.1 Stagnation State for Ideal Gas Model

#### 5.1.1 General Relationship

It is assumed that the flow is one–dimensional. Figure (5.1) describes a gas flow through a converging–diverging nozzle. It has been found that a theoretical state known as the stagnation state is very useful in simplifying the solution and treatment of the flow. The stagnation state is a theoretical state in which the flow is brought into a complete motionless condition in isentropic process without other forces (e.g. gravity force). Several properties that can be represented by this theoretical process which include temperature, pressure, and density et cetera and denoted by the subscript "0."

First, the stagnation temperature is calculated. The energy conservation can

be written as

$$h + \frac{U^2}{2} = h_0 \quad (5.1)$$

Perfect gas is an ideal gas with a constant heat capacity,  $C_p$ . For perfect gas equation (5.1) is simplified into

$$C_p T + \frac{U^2}{2} = C_p T_0 \quad (5.2)$$

Again it is common to denote  $T_0$  as the stagnation temperature. Recalling from thermodynamic the relationship for perfect gas

$$R = C_p - C_v \quad (5.3)$$

and denoting  $k \equiv C_p \div C_v$  then the thermodynamics relationship obtains the form

$$C_p = \frac{kR}{k-1} \quad (5.4)$$

and where  $R$  is a specific constant. Dividing equation (5.2) by  $(C_p T)$  yields

$$1 + \frac{U^2}{2C_p T} = \frac{T_0}{T} \quad (5.5)$$

Now, substituting  $c^2 = kRT$  or  $T = c^2/kR$  equation (5.5) changes into

$$1 + \frac{kRU^2}{2C_p c^2} = \frac{T_0}{T} \quad (5.6)$$

By utilizing the definition of  $k$  by equation (2.24) and inserting it into equation (5.6) yields

$$1 + \frac{k-1}{2} \frac{U^2}{c^2} = \frac{T_0}{T} \quad (5.7)$$

It very useful to convert equation (5.6) into a dimensionless form and denote Mach number as the ratio of velocity to speed of sound as

$$M \equiv \frac{U}{c} \quad (5.8)$$

Inserting the definition of Mach number (5.8) into equation (5.7) reads

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \quad (5.9)$$

The usefulness of Mach number and equation (5.9) can be demonstrated by this following simple example. In this example a gas flows through a tube (see Figure 5.2) of any shape can be expressed as a function of only the stagnation temperature as opposed to the function of the temperatures and velocities.

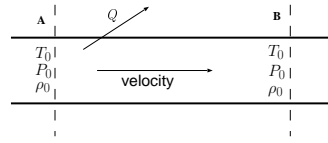


Fig. -5.2. Perfect gas flows through a tube

The definition of the stagnation state provides the advantage of compact writing. For example, writing the energy equation for the tube shown in Figure (5.2) can be reduced to

$$\dot{Q} = C_p(T_{0B} - T_{0A})\dot{m} \tag{5.10}$$

The ratio of stagnation pressure to the static pressure can be expressed as the function of the temperature ratio because of the isentropic relationship as

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} \tag{5.11}$$

In the same manner the relationship for the density ratio is

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{1}{k-1}} \tag{5.12}$$

A new useful definition is introduced for the case when  $M = 1$  and denoted by superscript “\*.” The special case of ratio of the star values to stagnation values are dependent only on the heat ratio as the following:

$$\frac{T^*}{T_0} = \frac{c^{*2}}{c_0^2} = \frac{2}{k+1} \tag{5.13}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{5.14}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \tag{5.15}$$

## Static Properties As A Function of Mach Number

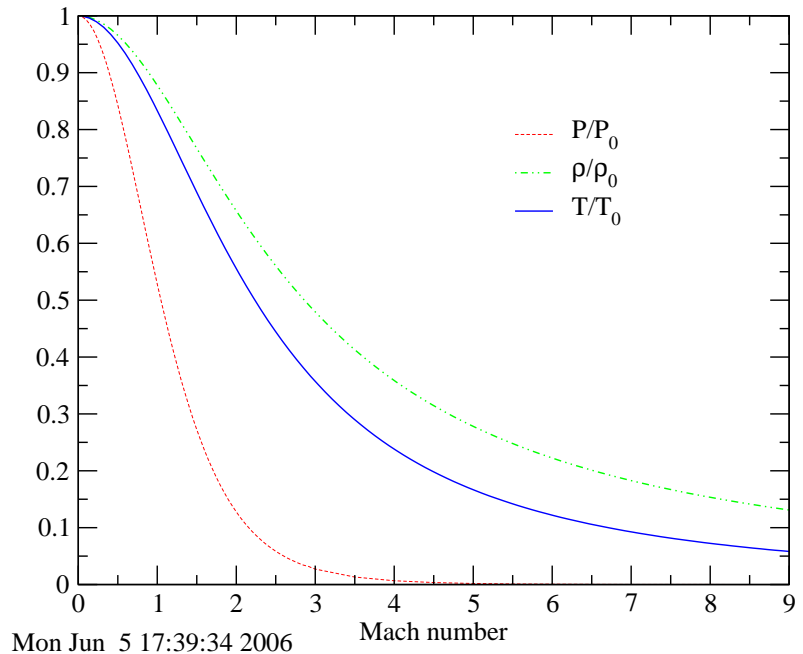


Fig. -5.3. The stagnation properties as a function of the Mach number,  $k = 1.4$

### 5.1.2 Relationships for Small Mach Number

Even with today's computers a simplified method can reduce the tedious work involved in computational work. In particular, the trends can be examined with analytical methods. It further will be used in the book to examine trends in derived models. It can be noticed that the Mach number involved in the above equations is in a square power. Hence, if an acceptable error is of about %1 then  $M < 0.1$  provides the desired range. Further, if a higher power is used, much smaller error results. First it can be noticed that the ratio of temperature to stagnation temperature,  $\frac{T}{T_0}$  is provided in power series. Expanding of the equations according to the binomial expansion of

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots \quad (5.16)$$

will result in the same fashion

$$\frac{P_0}{P} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48} \dots \quad (5.17)$$

$$\frac{\rho_0}{\rho} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48} \dots \quad (5.18)$$

The pressure difference normalized by the velocity (kinetic energy) as correction factor is

$$\frac{P_0 - P}{\frac{1}{2}\rho U^2} = 1 + \overbrace{\frac{M^2}{4} + \frac{(2-k)M^4}{24} + \dots}^{\text{compressibility correction}} \quad (5.19)$$

From the above equation, it can be observed that the correction factor approaches zero when  $M \rightarrow 0$  and then equation (5.19) approaches the standard equation for incompressible flow.

The definition of the star Mach is ratio of the velocity and star speed of sound at  $M = 1$ .

$$M^* = \frac{U}{c^*} = \sqrt{\frac{k+1}{2}} M \left( 1 - \frac{k-1}{4} M^2 + \dots \right) \quad (5.20)$$

$$\frac{P_0 - P}{P} = \frac{kM^2}{2} \left( 1 + \frac{M^2}{4} + \dots \right) \quad (5.21)$$

$$\frac{\rho_0 - \rho}{\rho} = \frac{M^2}{2} \left( 1 - \frac{kM^2}{4} + \dots \right) \quad (5.22)$$

The normalized mass rate becomes

$$\frac{\dot{m}}{A} = \sqrt{\frac{kP_0^2 M^2}{RT_0}} \left( 1 + \frac{k-1}{4} M^2 + \dots \right) \quad (5.23)$$

The ratio of the area to star area is

$$\frac{A}{A^*} = \left( \frac{2}{k+1} \right)^{\frac{k+1}{2(k-1)}} \left( \frac{1}{M} + \frac{k+1}{4} M + \frac{(3-k)(k+1)}{32} M^3 + \dots \right) \quad (5.24)$$

## 5.2 Isentropic Converging-Diverging Flow in Cross Section

The important sub case in this chapter is the flow in a converging–diverging nozzle. The control volume is shown in Figure (5.4). There are two models that assume variable area flow: First is isentropic and adiabatic model. Second is isentropic and isothermal model. Clearly, the stagnation temperature,  $T_0$ , is constant through the adiabatic flow because there isn't heat transfer. Therefore, the stagnation pressure is also constant through the flow because the flow isentropic. Conversely, in mathematical terms, equation (5.9) and equation (5.11) are the same. If the right hand side is constant for one variable, it is constant for the other. In the same argument, the stagnation density is constant through the flow. Thus, knowing the Mach number or the temperature will provide all that is needed to find the other properties. The only properties that need to be connected are the cross section area and the Mach number. Examination of the relation between properties can then be carried out.

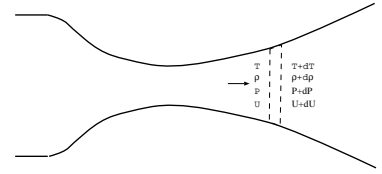


Fig. -5.4. Control volume inside a converging-diverging nozzle.

### 5.2.1 The Properties in the Adiabatic Nozzle

When there is no external work and heat transfer, the energy equation, reads

$$dh + U dU = 0 \quad (5.25)$$

Differentiation of continuity equation,  $\rho AU = \dot{m} = \text{constant}$ , and dividing by the continuity equation reads

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dU}{U} = 0 \quad (5.26)$$

The thermodynamic relationship between the properties can be expressed as

$$T ds = dh - \frac{dP}{\rho} \quad (5.27)$$

For isentropic process  $ds \equiv 0$  and combining equations (5.25) with (5.27) yields

$$\frac{dP}{\rho} + U dU = 0 \quad (5.28)$$

Differentiation of the equation state (perfect gas),  $P = \rho RT$ , and dividing the results by the equation of state ( $\rho RT$ ) yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (5.29)$$

Obtaining an expression for  $dU/U$  from the mass balance equation (5.26) and using it in equation (5.28) reads

$$\frac{dP}{\rho} - U^2 \overbrace{\left[ \frac{dA}{A} + \frac{d\rho}{\rho} \right]}^{\frac{dU}{U}} = 0 \quad (5.30)$$

Rearranging equation (5.30) so that the density,  $\rho$ , can be replaced by the static pressure,  $dP/\rho$  yields

$$\frac{dP}{\rho} = U^2 \left( \frac{dA}{A} + \frac{d\rho}{\rho} \frac{dP}{dP} \right) = U^2 \left( \frac{dA}{A} + \overbrace{\frac{d\rho}{dP}}^{\frac{1}{c^2}} \frac{dP}{\rho} \right) \quad (5.31)$$

Recalling that  $dP/d\rho = c^2$  and substitute the speed of sound into equation (5.31) to obtain

$$\frac{dP}{\rho} \left[ 1 - \left( \frac{U}{c} \right)^2 \right] = U^2 \frac{dA}{A} \quad (5.32)$$

Or in a dimensionless form

$$\frac{dP}{\rho} (1 - M^2) = U^2 \frac{dA}{A} \quad (5.33)$$

Equation (5.33) is a differential equation for the pressure as a function of the cross section area. It is convenient to rearrange equation (5.33) to obtain a variables separation form of

$$dP = \frac{\rho U^2}{A} \frac{dA}{1 - M^2} \quad (5.34)$$

### The pressure Mach number relationship

Before going further in the mathematical derivation it is worth looking at the physical meaning of equation (5.34). The term  $\rho U^2/A$  is always positive (because all the three terms can be only positive). Now, it can be observed that  $dP$  can be positive or negative depending on the  $dA$  and Mach number. The meaning of the sign change for the pressure differential is that the pressure can increase or decrease. It can be observed that the critical Mach number is one. If the Mach number is larger than one than  $dP$  has opposite sign of  $dA$ . If Mach number is smaller than one  $dP$  and  $dA$  have the same sign. For the subsonic branch  $M < 1$  the term  $1/(1 - M^2)$  is positive hence

$$\begin{aligned} dA > 0 &\implies dP > 0 \\ dA < 0 &\implies dP < 0 \end{aligned}$$

From these observations the trends are similar to those in incompressible fluid. An increase in area results in an increase of the static pressure (converting the dynamic pressure to a static pressure). Conversely, if the area decreases (as a function of  $x$ ) the pressure decreases. Note that the pressure decrease is larger in compressible flow compared to incompressible flow.

For the supersonic branch  $M > 1$ , the phenomenon is different. For  $M > 1$  the term  $1/1 - M^2$  is negative and change the character of the equation.

$$\begin{aligned} dA > 0 &\Rightarrow dP < 0 \\ dA < 0 &\Rightarrow dP > 0 \end{aligned}$$

This behavior is opposite to incompressible flow behavior.

For the special case of  $M = 1$  (sonic flow) the value of the term  $1 - M^2 = 0$  thus mathematically  $dP \rightarrow \infty$  or  $dA = 0$ . Since physically  $dP$  can increase only in a finite amount it must that  $dA = 0$ . It must also be noted that when  $M = 1$  occurs only when  $dA = 0$ . However, the opposite, not necessarily means that when  $dA = 0$  that  $M = 1$ . In that case, it is possible that  $dM = 0$  thus the diverging side is in the subsonic branch and the flow isn't choked.

The relationship between the velocity and the pressure can be observed from equation (5.28) by solving it for  $dU$ .

$$dU = -\frac{dP}{PU} \quad (5.35)$$

From equation (5.35) it is obvious that  $dU$  has an opposite sign to  $dP$  (since the term  $PU$  is positive). Hence the pressure increases when the velocity decreases and vice versa.

From the speed of sound, one can observe that the density,  $\rho$ , increases with pressure and vice versa (see equation 5.36).

$$d\rho = \frac{1}{c^2} dP \quad (5.36)$$

It can be noted that in the derivations of the above equations (5.35 - 5.36), the equation of state was not used. Thus, the equations are applicable for any gas (perfect or imperfect gas).

The second law (isentropic relationship) dictates that  $ds = 0$  and from thermodynamics

$$ds = 0 = C_p \frac{dT}{T} - R \frac{dP}{P}$$

and for perfect gas

$$\frac{dT}{T} = \frac{k-1}{k} \frac{dP}{P} \quad (5.37)$$

Thus, the temperature varies according to the same way that pressure does.

The relationship between the Mach number and the temperature can be obtained by utilizing the fact that the process is assumed to be adiabatic  $dT_0 = 0$ . Differentiation of equation (5.9), the relationship between the temperature and the stagnation temperature becomes

$$dT_0 = 0 = dT \left( 1 + \frac{k-1}{2} M^2 \right) + T(k-1)M dM \quad (5.38)$$

and simplifying equation (5.38) yields

$$\frac{dT}{T} = - \frac{(k-1)M dM}{1 + \frac{k-1}{2} M^2} \quad (5.39)$$

### Relationship Between the Mach Number and Cross Section Area

The equations used in the solution are energy (5.39), second law (5.37), state (5.29), mass (5.26)<sup>1</sup>. Note, equation (5.33) isn't the solution but demonstration of certain properties on the pressure.

The relationship between temperature and the cross section area can be obtained by utilizing the relationship between the pressure and temperature (5.37) and the relationship of pressure and cross section area (5.33). First stage equation (5.39) is combined with equation (5.37) and becomes

$$\frac{(k-1)}{k} \frac{dP}{P} = - \frac{(k-1)M dM}{1 + \frac{k-1}{2} M^2} \quad (5.40)$$

Combining equation (5.40) with equation (5.33) yields

$$\frac{1}{k} \frac{\rho U^2}{A} \frac{dA}{1-M^2} = - \frac{M dM}{1 + \frac{k-1}{2} M^2} \quad (5.41)$$

The following identity,  $\rho U^2 = kMP$  can be proved as

$$kM^2 P = k \overbrace{\frac{U^2}{c^2}}^{M^2} \overbrace{\rho RT}^P = k \frac{U^2}{kRT} \overbrace{\rho RT}^P = \rho U^2 \quad (5.42)$$

Using the identity in equation (5.42) changes equation (5.41) into

$$\frac{dA}{A} = \frac{M^2 - 1}{M \left( 1 + \frac{k-1}{2} M^2 \right)} dM \quad (5.43)$$

<sup>1</sup>The momentum equation is not used normally in isentropic process, why?

Equation (5.43) is very important because it relates the geometry (area) with the relative velocity (Mach number). In equation (5.43), the factors  $M \left(1 + \frac{k-1}{2} M^2\right)$  and  $A$  are positive regardless of the values of  $M$  or  $A$ . Therefore, the only factor that affects relationship between the cross area and the Mach number is  $M^2 - 1$ . For  $M < 1$  the Mach number is varied opposite to the cross section area. In the case of  $M > 1$  the Mach number increases with the cross section area and vice versa. The special case is when  $M = 1$  which requires that  $dA = 0$ . This condition imposes that internal<sup>2</sup> flow has to pass a converging-diverging device to obtain supersonic velocity. This minimum area is referred to as "throat."

Again, the opposite conclusion that when  $dA = 0$  implies that  $M = 1$  is not correct because possibility of  $dM = 0$ . In subsonic flow branch, from the mathematical point of view: on one hand, a decrease of the cross section increases the velocity and the Mach number, on the other hand, an increase of the cross section decreases the velocity and Mach number (see Figure (5.5)).

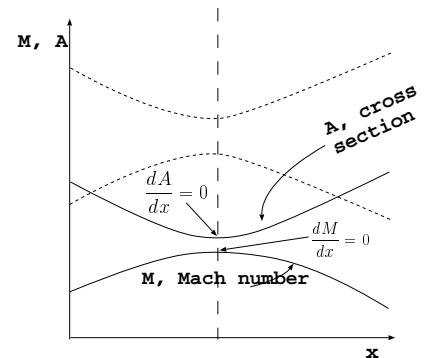


Fig. -5.5. The relationship between the cross section and the Mach number on the subsonic branch

## 5.2.2 Isentropic Flow Examples

Example 5.1:

Air is allowed to flow from a reservoir with temperature of  $21^\circ\text{C}$  and with pressure of  $5[\text{MPa}]$  through a tube. It was measured that air mass flow rate is  $1[\text{kg}/\text{sec}]$ . At some point on the tube static pressure was measured to be  $3[\text{MPa}]$ . Assume that process is isentropic and neglect the velocity at the reservoir, calculate the Mach number, velocity, and the cross section area at that point where the static pressure was measured. Assume that the ratio of specific heat is  $k = C_p/C_v = 1.4$ .

### SOLUTION

The stagnation conditions at the reservoir will be maintained throughout the tube because the process is isentropic. Hence the stagnation temperature can be written  $T_0 = \text{constant}$  and  $P_0 = \text{constant}$  and both of them are known (the condition at the reservoir). For the point where the static pressure is known, the Mach number can be calculated by utilizing the pressure ratio. With the known Mach number, the temperature, and velocity can be calculated. Finally, the cross section can be calculated

<sup>2</sup>This condition does not impose any restrictions for external flow. In external flow, an object can be moved in arbitrary speed.

with all these information.

In the point where the static pressure known

$$\bar{P} = \frac{P}{P_0} = \frac{3[MPa]}{5[MPa]} = 0.6$$

From Table (5.2) or from Figure (5.3) or utilizing the enclosed program, Potto-GDC, or simply using the equations shows that

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.88639	0.86420	0.69428	1.0115	0.60000	0.60693	0.53105

With these values the static temperature and the density can be calculated.

$$T = 0.86420338 \times (273 + 21) = 254.076K$$

$$\rho = \frac{\rho}{\rho_0} \overbrace{\frac{P_0}{RT_0}}^{\rho_0} = 0.69428839 \times \frac{5 \times 10^6 [Pa]}{287.0 \left[ \frac{J}{kgK} \right] \times 294 [K]}$$

$$= 41.1416 \left[ \frac{kg}{m^3} \right]$$

The velocity at that point is

$$U = M \overbrace{\sqrt{kRT}}^c = 0.88638317 \times \sqrt{1.4 \times 287 \times 294} = 304[m/sec]$$

The tube area can be obtained from the mass conservation as

$$A = \frac{\dot{m}}{\rho U} = 8.26 \times 10^{-5} [m^3]$$

For a circular tube the diameter is about 1[cm].

---

End solution

**Example 5.2:**

The Mach number at point A on tube is measured to be  $M = 2^3$  and the static pressure is  $2[Bar]^4$ . Downstream at point B the pressure was measured to be  $1.5[Bar]$ . Calculate the Mach number at point B under the isentropic flow assumption. Also, estimate the temperature at point B. Assume that the specific heat ratio  $k = 1.4$  and assume a perfect gas model.

<sup>4</sup>This pressure is about two atmospheres with temperature of  $250[K]$

<sup>4</sup>Well, this question is for academic purposes, there is no known way for the author to directly measure the Mach number. The best approximation is by using inserted cone for supersonic flow and measure the oblique shock. Here it is subsonic and this technique is not suitable.

SOLUTION

With the known Mach number at point A all the ratios of the static properties to total (stagnation) properties can be calculated. Therefore, the stagnation pressure at point A is known and stagnation temperature can be calculated.

At  $M = 2$  (supersonic flow) the ratios are

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
2.0000	0.555556	0.23005	1.6875	0.12780	0.21567	0.59309

With this information the pressure at point B can be expressed as

from the table

5.2 @  $M = 2$

$$\frac{P_A}{P_0} = \underbrace{\frac{P_B}{P_0}}_{M=2} \times \frac{P_A}{P_B} = 0.12780453 \times \frac{2.0}{1.5} = 0.17040604$$

The corresponding Mach number for this pressure ratio is 1.8137788 and  $T_B = 0.60315132 \frac{P_B}{P_0} = 0.17040879$ . The stagnation temperature can be "bypassed" to calculate the temperature at point B

$$T_B = T_A \times \underbrace{\frac{T_0}{T_A}}_{M=2} \times \underbrace{\frac{T_B}{T_0}}_{M=1.81..} = 250[K] \times \frac{1}{0.55555556} \times 0.60315132 \approx 271.42[K]$$

---

End solution

---

**Example 5.3:**

*Gas flows through a converging-diverging duct. At point "A" the cross section area is 50 [cm<sup>2</sup>] and the Mach number was measured to be 0.4. At point B in the duct the cross section area is 40 [cm<sup>2</sup>]. Find the Mach number at point B. Assume that the flow is isentropic and the gas specific heat ratio is 1.4.*

SOLUTION

To obtain the Mach number at point B by finding the ratio of the area to the critical area. This relationship can be obtained by

$$\frac{A_B}{A^*} = \frac{A_B}{A_A} \times \frac{A_A}{A^*} = \frac{40}{50} \times \underbrace{\frac{A_A}{A^*}}_{\text{from the Table 5.2}} = 1.272112$$

With the value of  $\frac{A_B}{A^*}$  from the Table (5.2) or from Potto-GDC two solutions can be obtained. The two possible solutions: the first supersonic  $M = 1.6265306$  and second subsonic  $M = 0.53884934$ . Both solution are possible and acceptable. The supersonic branch solution is possible only if there where a transition at throat where  $M=1$ .

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$
1.6266	0.65396	0.34585	1.2721	0.22617	0.28772
0.53887	0.94511	0.86838	1.2721	0.82071	1.0440

End solution

**Example 5.4:**

Engineer needs to redesign a syringe for medical applications. The complain in the syringe is that the syringe is “hard to push.” The engineer analyzes the flow and conclude that the flow is choke. Upon this fact, what engineer should do with syringe increase the pushing diameter or decrease the diameter? Explain.

SOLUTION

This problem is a typical to compressible flow in the sense the solution is opposite the regular intuition. The diameter should be decreased. The pressure in the choke flow in the syringe is past the critical pressure ratio. Hence, the force is a function of the cross area of the syringe. So, to decrease the force one should decrease the area.

End solution

**5.2.3 Mass Flow Rate (Number)**

One of the important engineering parameters is the mass flow rate which for ideal gas is

$$\dot{m} = \rho U A = \frac{P}{RT} U A \tag{5.44}$$

This parameter is studied here, to examine the maximum flow rate and to see what is the effect of the compressibility on the flow rate. The area ratio as a function of the Mach number needed to be established, specifically and explicitly the relationship for the choked flow. The area ratio is defined as the ratio of the cross section at any point to the throat area (the narrow area). It is convenient to rearrange the equation (5.44) to be expressed in terms of the stagnation properties as

$$\frac{\dot{m}}{A} = \frac{P}{P_0} \frac{P_0 U}{\sqrt{kRT}} \sqrt{\frac{k}{R}} \sqrt{\frac{T_0}{T}} \frac{1}{\sqrt{T_0}} = \frac{P_0}{\sqrt{T_0}} M \sqrt{\frac{k}{R}} \frac{P}{P_0} \sqrt{\frac{T_0}{T}} \tag{5.45}$$

$f(M,k)$

Expressing the temperature in terms of Mach number in equation (5.45) results in

$$\frac{\dot{m}}{A} = \left( \frac{kMP_0}{\sqrt{kRT_0}} \right) \left( 1 + \frac{k-1}{2} M^2 \right)^{-\frac{k+1}{2(k-1)}} \tag{5.46}$$

It can be noted that equation (5.46) holds everywhere in the converging-diverging duct and this statement also true for the throat. The throat area can be denoted as

by  $A^*$ . It can be noticed that at the throat when the flow is choked or in other words  $M = 1$  and that the stagnation conditions (i.e. temperature, pressure) do not change. Hence equation (5.46) obtained the form

$$\frac{\dot{m}}{A^*} = \left( \frac{\sqrt{k}P_0}{\sqrt{RT_0}} \right) \left( 1 + \frac{k-1}{2} \right)^{-\frac{k+1}{2(k-1)}} \quad (5.47)$$

Since the mass flow rate is constant in the duct, dividing equations (5.47) by equation (5.46) yields

$$\frac{A}{A^*} = \frac{1}{M} \left( \frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}} \right)^{\frac{k+1}{2(k-1)}} \quad (5.48)$$

Equation (5.48) relates the Mach number at any point to the cross section area ratio.

The maximum flow rate can be expressed either by taking the derivative of equation (5.47) in with respect to  $M$  and equating to zero. Carrying this calculation results at  $M = 1$ .

$$\left( \frac{\dot{m}}{A^*} \right)_{max} \frac{P_0}{\sqrt{T_0}} = \sqrt{\frac{k}{R}} \left( \frac{k+1}{2} \right)^{-\frac{k+1}{2(k-1)}} \quad (5.49)$$

For specific heat ratio,  $k = 1.4$

$$\left( \frac{\dot{m}}{A^*} \right)_{max} \frac{P_0}{\sqrt{T_0}} \sim \frac{0.68473}{\sqrt{R}} \quad (5.50)$$

The maximum flow rate for air ( $R = 287 \text{ J/kgK}$ ) becomes,

$$\frac{\dot{m}\sqrt{T_0}}{A^*P_0} = 0.040418 \quad (5.51)$$

Equation (5.51) is known as Fliegner's Formula on the name of one of the first engineers who observed experimentally the choking phenomenon. It can be noticed that Fliegner's equation can lead to definition of the Fliegner's Number.

$$\frac{\dot{m}\sqrt{T_0}}{A^*P_0} = \frac{\dot{m} \overbrace{\sqrt{kRT_0}}^{c_0}}{\sqrt{kRA^*P_0}} = \frac{\overbrace{\dot{m}c_0}^{Fn}}{\sqrt{RA^*P_0}} \frac{1}{\sqrt{k}} \quad (5.52)$$

The definition of Fliegner's number ( $Fn$ ) is

$$Fn \equiv \frac{\dot{m}c_0}{\sqrt{RA^*P_0}} \quad (5.53)$$

## 5.2. ISENTROPIC CONVERGING-DIVERGING FLOW IN CROSS SECTION 67

Utilizing Fliengner's number definition and substituting it into equation (5.47) results in

$$Fn = kM \left( 1 + \frac{k-1}{2} M^2 \right)^{-\frac{k+1}{2(k-1)}} \quad (5.54)$$

and the maximum point for  $Fn$  at  $M = 1$  is

$$Fn = k \left( \frac{k+1}{2} \right)^{-\frac{k+1}{2(k-1)}} \quad (5.55)$$

### “Naughty Professor” Problems in Isentropic Flow

To explain the material better some instructors invented problems, which have mostly academic purpose, (see for example, Shapiro (problem 4.5)). While these problems have a limit applicability in reality, they have substantial academic value and therefore presented here. The situation where the mass flow rate per area given with one of the stagnation properties and one of the static properties, e.g.  $P_0$  and  $T$  or  $T_0$  and  $P$  present difficulty for the calculations. The use of the regular isentropic Table is not possible because there isn't variable represent this kind problems. For this kind of problems a new Table was constructed and present here<sup>5</sup>.

#### The case of $T_0$ and $P$

This case considered to be simplest case and will first presented here. Using energy equation (5.9) and substituting for Mach number  $M = \dot{m}/A\rho c$  results in

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} \left( \frac{\dot{m}}{A\rho c} \right)^2 \quad (5.56)$$

Rearranging equation (5.56) result in

$$T_0\rho^2 = \overbrace{\frac{P}{T\rho}}^{\frac{P}{T\rho}} \rho + \overbrace{\left( \frac{T}{c^2} \right)^{1/kR} \frac{k-1}{2}}^{\frac{1}{kR}} \left( \frac{\dot{m}}{A} \right)^2 \quad (5.57)$$

And further Rearranging equation (5.57) transformed it into

$$\rho^2 = \frac{P\rho}{T_0R} + \frac{k-1}{2kRT_0} \left( \frac{\dot{m}}{A} \right)^2 \quad (5.58)$$

Equation (5.58) is quadratic equation for density,  $\rho$  when all other variables are known. It is convenient to change it into

$$\rho^2 - \frac{P\rho}{T_0R} - \frac{k-1}{2kRT_0} \left( \frac{\dot{m}}{A} \right)^2 = 0 \quad (5.59)$$

<sup>5</sup>Since version 0.44 of this book.

The only physical solution is when the density is positive and thus the only solution is

$$\rho = \frac{1}{2} \left[ \frac{P}{RT_0} + \sqrt{\left(\frac{P}{RT_0}\right)^2 + 2 \underbrace{\frac{k-1}{kRT_0} \left(\frac{\dot{m}}{A}\right)^2}_{\leftarrow (M \rightarrow 0) \rightarrow 0}} \right] \quad (5.60)$$

For almost incompressible flow the density is reduced and the familiar form of perfect gas model is seen since stagnation temperature is approaching the static temperature for very small Mach number ( $\rho = \frac{P}{RT_0}$ ). In other words, the terms for the group over the under-brace approaches zero when the flow rate (Mach number) is very small.

It is convenient to denote a new dimensionless density as

$$\hat{\rho} = \frac{\rho}{\frac{P}{RT_0}} = \frac{\rho RT_0}{P} = \frac{1}{T} \quad (5.61)$$

With this new definition equation (5.60) is transformed into

$$\hat{\rho} = \frac{1}{2} \left( 1 + \sqrt{1 + 2 \frac{(k-1)RT_0}{kP^2} \left(\frac{\dot{m}}{A}\right)^2} \right) \quad (5.62)$$

The dimensionless density now is related to a dimensionless group that is a function of Fn number and Mach number only! Thus, this dimensionless group is function of Mach number only.

$$\frac{RT_0}{P^2} \left(\frac{\dot{m}}{A}\right)^2 = \frac{1}{k} \overbrace{\frac{c_0^2}{P_0^2} \left(\frac{\dot{m}}{A^*}\right)^2}^{Fn^2} \overbrace{\left(\frac{A^*}{A}\right)^2 \left(\frac{P_0}{P}\right)^2}^{\frac{A^*P_0}{AP} = f(M)} \quad (5.63)$$

Thus,

$$\frac{RT_0}{P^2} \left(\frac{\dot{m}}{A}\right)^2 = \frac{Fn^2}{k} \left(\frac{A^*P_0}{AP}\right)^2 \quad (5.64)$$

Hence, the dimensionless density is

$$\hat{\rho} = \frac{1}{2} \left( 1 + \sqrt{1 + 2 \frac{(k-1)Fn^2}{k^2} \left(\frac{A^*P_0}{AP}\right)^2} \right) \quad (5.65)$$

Again notice that the right hand side of equation (5.65) is only function of Mach number (well, also the specific heat,  $k$ ). And the values of  $\frac{AP}{A^*P_0}$  were tabulated in Table (5.2) and Fn is tabulated in the next Table (5.1). Thus, the problems is reduced to finding tabulated values.

**The case of  $P_0$  and  $T$** 

A similar problem can be described for the case of stagnation pressure,  $P_0$ , and static temperature,  $T$ .

First, it is shown that the dimensionless group is a function of Mach number only (well, again the specific heat ratio,  $k$  also).

$$\frac{RT}{P_0^2} \left( \frac{\dot{m}}{A} \right)^2 = \frac{Fn^2}{k} \left( \frac{A^* P_0}{AP} \right)^2 \left( \frac{T}{T_0} \right) \left( \frac{P_0}{P} \right)^2 \quad (5.66)$$

It can be noticed that

$$\frac{Fn^2}{k} = \left( \frac{T}{T_0} \right) \left( \frac{P_0}{P} \right)^2 \quad (5.67)$$

Thus equation (5.66) became

$$\frac{RT}{P_0^2} \left( \frac{\dot{m}}{A} \right)^2 = \left( \frac{A^* P_0}{AP} \right)^2 \quad (5.68)$$

The right hand side is tabulated in the “regular” isentropic Table such (5.2). This example shows how a dimensional analysis is used to solve a problems without actually solving any equations. The actual solution of the equation is left as exercise (this example under construction). What is the legitimacy of this method? The explanation simply based the previous experience in which for a given ratio of area or pressure ratio (etcetera) determines the Mach number. Based on the same arguments, if it was shown that a group of parameters depends only Mach number than the Mach is determined by this group.

The method of solution for given these parameters is by calculating the  $\frac{PA}{P_0 A^*}$  and then using the table to find the corresponding Mach number.

**The case of  $\rho_0$  and  $T$  or  $P$** 

The last case sometimes referred to as the “naughty professor’s question” case dealt here is when the stagnation density given with the static temperature/pressure. First, the dimensionless approach is used and later analytical method is discussed (under construction).

$$\frac{1}{R\rho_0 P} \left( \frac{\dot{m}}{A} \right)^2 = \frac{\overbrace{kRT_0}^{c_0^2}}{kRP_0 P_0 \frac{P}{P_0}} \left( \frac{\dot{m}}{A} \right)^2 = \frac{c_0^2}{kRP_0^2 \frac{P}{P_0}} \left( \frac{\dot{m}}{A} \right)^2 = \frac{Fn^2}{k} \left( \frac{P_0}{P} \right) \quad (5.69)$$

The last case dealt here is of the stagnation density with static pressure and the following is dimensionless group

$$\frac{1}{R\rho_0^2 T} \left( \frac{\dot{m}}{A} \right)^2 = \frac{\overbrace{kRT_0 T_0}^{c_0^2}}{kRP_0^2 T} \left( \frac{\dot{m}}{A} \right)^2 = \frac{c_0^2 T_0}{kRP_0^2 T} \left( \frac{\dot{m}}{A} \right)^2 = \frac{Fn^2}{k} \left( \frac{T_0}{T} \right) \quad (5.70)$$

It was hidden in the derivations/explanations of the above analysis didn't explicitly state under what conditions these analysis is correct. Unfortunately, not all the analysis valid for the same conditions and is as the regular "isentropic" Table, (5.2). The heat/temperature part is valid for **enough** adiabatic condition while the pressure condition requires also isentropic process. All the above conditions/situations require to have the perfect gas model as the equation of state. For example the first "naughty professor" question is sufficient that process is adiabatic only ( $T_0$ ,  $P$ , mass flow rate per area.).

Table -5.1. Fliegner's number and other parameters as a function of Mach number

M	F <sub>n</sub>	$\hat{\rho}$	$\left(\frac{P_0 A^*}{AP}\right)^2$	$\frac{RT_0}{P^2} \left(\frac{\dot{m}}{A}\right)^2$	$\frac{1}{R\rho_0 P} \left(\frac{\dot{m}}{A}\right)^2$	$\frac{1}{R\rho_0^2 T} \left(\frac{\dot{m}}{A}\right)^2$
0.00E+00	1.40E-06	1.000	0.0	0.0	0.0	0.0
0.05000	0.070106	1.000	0.00747	2.62E-05	0.00352	0.00351
0.10000	0.14084	1.000	0.029920	0.000424	0.014268	0.014197
0.20000	0.28677	1.001	0.12039	0.00707	0.060404	0.059212
0.21000	0.30185	1.001	0.13284	0.00865	0.067111	0.065654
0.22000	0.31703	1.001	0.14592	0.010476	0.074254	0.072487
0.23000	0.33233	1.002	0.15963	0.012593	0.081847	0.079722
0.24000	0.34775	1.002	0.17397	0.015027	0.089910	0.087372
0.25000	0.36329	1.003	0.18896	0.017813	0.098460	0.095449
0.26000	0.37896	1.003	0.20458	0.020986	0.10752	0.10397
0.27000	0.39478	1.003	0.22085	0.024585	0.11710	0.11294
0.28000	0.41073	1.004	0.23777	0.028651	0.12724	0.12239
0.29000	0.42683	1.005	0.25535	0.033229	0.13796	0.13232
0.30000	0.44309	1.005	0.27358	0.038365	0.14927	0.14276
0.31000	0.45951	1.006	0.29247	0.044110	0.16121	0.15372
0.32000	0.47609	1.007	0.31203	0.050518	0.17381	0.16522
0.33000	0.49285	1.008	0.33226	0.057647	0.18709	0.17728
0.34000	0.50978	1.009	0.35316	0.065557	0.20109	0.18992
0.35000	0.52690	1.011	0.37474	0.074314	0.21584	0.20316
0.36000	0.54422	1.012	0.39701	0.083989	0.23137	0.21703
0.37000	0.56172	1.013	0.41997	0.094654	0.24773	0.23155
0.38000	0.57944	1.015	0.44363	0.10639	0.26495	0.24674
0.39000	0.59736	1.017	0.46798	0.11928	0.28307	0.26264
0.40000	0.61550	1.019	0.49305	0.13342	0.30214	0.27926
0.41000	0.63386	1.021	0.51882	0.14889	0.32220	0.29663
0.42000	0.65246	1.023	0.54531	0.16581	0.34330	0.31480
0.43000	0.67129	1.026	0.57253	0.18428	0.36550	0.33378
0.44000	0.69036	1.028	0.60047	0.20442	0.38884	0.35361
0.45000	0.70969	1.031	0.62915	0.22634	0.41338	0.37432
0.46000	0.72927	1.035	0.65857	0.25018	0.43919	0.39596

Table -5.1. Fliegner's number and other parameters as function of Mach number (continue)

M	F <sub>n</sub>	$\hat{\rho}$	$\left(\frac{P_0 A^*}{AP}\right)^2$	$\frac{RT_0}{P^2} \left(\frac{\dot{m}}{A}\right)^2$	$\frac{1}{R\rho_0 P} \left(\frac{\dot{m}}{A}\right)^2$	$\frac{1}{R\rho_0^2 T} \left(\frac{\dot{m}}{A}\right)^2$
0.47000	0.74912	1.038	0.68875	0.27608	0.46633	0.41855
0.48000	0.76924	1.042	0.71967	0.30418	0.49485	0.44215
0.49000	0.78965	1.046	0.75136	0.33465	0.52485	0.46677
0.50000	0.81034	1.050	0.78382	0.36764	0.55637	0.49249
0.51000	0.83132	1.055	0.81706	0.40333	0.58952	0.51932
0.52000	0.85261	1.060	0.85107	0.44192	0.62436	0.54733
0.53000	0.87421	1.065	0.88588	0.48360	0.66098	0.57656
0.54000	0.89613	1.071	0.92149	0.52858	0.69948	0.60706
0.55000	0.91838	1.077	0.95791	0.57709	0.73995	0.63889
0.56000	0.94096	1.083	0.99514	0.62936	0.78250	0.67210
0.57000	0.96389	1.090	1.033	0.68565	0.82722	0.70675
0.58000	0.98717	1.097	1.072	0.74624	0.87424	0.74290
0.59000	1.011	1.105	1.112	0.81139	0.92366	0.78062
0.60000	1.035	1.113	1.152	0.88142	0.97562	0.81996
0.61000	1.059	1.122	1.194	0.95665	1.030	0.86101
0.62000	1.084	1.131	1.236	1.037	1.088	0.90382
0.63000	1.109	1.141	1.279	1.124	1.148	0.94848
0.64000	1.135	1.151	1.323	1.217	1.212	0.99507
0.65000	1.161	1.162	1.368	1.317	1.278	1.044
0.66000	1.187	1.173	1.414	1.423	1.349	1.094
0.67000	1.214	1.185	1.461	1.538	1.422	1.147
0.68000	1.241	1.198	1.508	1.660	1.500	1.202
0.69000	1.269	1.211	1.557	1.791	1.582	1.260
0.70000	1.297	1.225	1.607	1.931	1.667	1.320
0.71000	1.326	1.240	1.657	2.081	1.758	1.382
0.72000	1.355	1.255	1.708	2.241	1.853	1.448
0.73000	1.385	1.271	1.761	2.412	1.953	1.516
0.74000	1.415	1.288	1.814	2.595	2.058	1.587
0.75000	1.446	1.305	1.869	2.790	2.168	1.661
0.76000	1.477	1.324	1.924	2.998	2.284	1.738
0.77000	1.509	1.343	1.980	3.220	2.407	1.819
0.78000	1.541	1.362	2.038	3.457	2.536	1.903
0.79000	1.574	1.383	2.096	3.709	2.671	1.991
0.80000	1.607	1.405	2.156	3.979	2.813	2.082
0.81000	1.642	1.427	2.216	4.266	2.963	2.177
0.82000	1.676	1.450	2.278	4.571	3.121	2.277
0.83000	1.712	1.474	2.340	4.897	3.287	2.381
0.84000	1.747	1.500	2.404	5.244	3.462	2.489
0.85000	1.784	1.526	2.469	5.613	3.646	2.602

Table -5.1. Fliegner's number and other parameters as function of Mach number (continue)

M	Fn	$\hat{\rho}$	$\left(\frac{P_0 A^*}{AP}\right)^2$	$\frac{RT_0}{P^2} \left(\frac{\dot{m}}{A}\right)^2$	$\frac{1}{R\rho_0 P} \left(\frac{\dot{m}}{A}\right)^2$	$\frac{1}{R\rho_0^2 T} \left(\frac{\dot{m}}{A}\right)^2$
0.86000	1.821	1.553	2.535	6.006	3.840	2.720
0.87000	1.859	1.581	2.602	6.424	4.043	2.842
0.88000	1.898	1.610	2.670	6.869	4.258	2.971
0.89000	1.937	1.640	2.740	7.342	4.484	3.104
0.90000	1.977	1.671	2.810	7.846	4.721	3.244
0.91000	2.018	1.703	2.882	8.381	4.972	3.389
0.92000	2.059	1.736	2.955	8.949	5.235	3.541
0.93000	2.101	1.771	3.029	9.554	5.513	3.699
0.94000	2.144	1.806	3.105	10.20	5.805	3.865
0.95000	2.188	1.843	3.181	10.88	6.112	4.037
0.96000	2.233	1.881	3.259	11.60	6.436	4.217
0.97000	2.278	1.920	3.338	12.37	6.777	4.404
0.98000	2.324	1.961	3.419	13.19	7.136	4.600
0.99000	2.371	2.003	3.500	14.06	7.515	4.804
1.000	2.419	2.046	3.583	14.98	7.913	5.016

## Example 5.5:

A gas flows in the tube with mass flow rate of 0.1 [kg/sec] and tube cross section is 0.001[m<sup>2</sup>]. The temperature at Chamber supplying the pressure to tube is 27°C. At some point the static pressure was measured to be 1.5[Bar]. Calculate for that point the Mach number, the velocity, and the stagnation pressure. Assume that the process is isentropic,  $k = 1.3$ ,  $R = 287$ [j/kgK].

## SOLUTION

The first thing that need to be done is to find the mass flow per area and it is

$$\frac{\dot{m}}{A} = 0.1/0.001 = 100.0[\text{kg}/\text{sec}/\text{m}^2]$$

It can be noticed that the total temperature is 300K and the static pressure is 1.5[Bar]. The solution is based on section equations (5.60) through (5.65). It is fortunate that Potto-GDC exist and it can be just plug into it and it provide that

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.17124	0.99562	0.98548	3.4757	0.98116	3.4102	1.5392

The velocity can be calculated as

$$U = Mc = \sqrt{kRTM} = 0.17 \times \sqrt{1.3 \times 287 \times 300} \times \sim 56.87[\text{m}/\text{sec}]$$

The stagnation pressure is

$$P_0 = \frac{P}{P/P_0} = 1.5/0.98116 = 1.5288[\text{Bar}]$$

---

End solution

---

### Flow with pressure losses

The expression for the mass flow rate (5.46) is appropriate regardless the flow is isentropic or adiabatic. That expression was derived based on the theoretical total pressure and temperature (Mach number) which does not based on the considerations whether the flow is isentropic or adiabatic. In the same manner the definition of  $A^*$  referred to the theoretical minimum area ("throat area") if the flow continues to flow in an isentropic manner. Clearly, in a case where the flow isn't isentropic or adiabatic the total pressure and the total temperature will change (due to friction, and heat transfer). A constant flow rate requires that  $\dot{m}_A = \dot{m}_B$ . Denoting subscript A for one point and subscript B for another point mass equation (5.47) can be equated as

$$\left( \frac{kP_0A^*}{RT_0} \right) \left( 1 + \frac{k-1}{2}M^2 \right)^{-\frac{k-1}{2(k-1)}} = \text{constant} \quad (5.71)$$

From equation (5.71), it is clear that the function  $f(P_0, T_0, A^*) = \text{constant}$ . There are two possible models that can be used to simplify the calculations. The first model for neglected heat transfer (adiabatic) flow and in which the total temperature remained constant (Fanno flow like). The second model which there is significant heat transfer but insignificant pressure loss (Rayleigh flow like).

If the mass flow rate is constant at any point on the tube (no mass loss occur) then

$$\dot{m} = A^* \sqrt{\frac{k}{RT_0} \left( \frac{2}{k+1} \right)^{\frac{k+1}{k-1}}} P_0 \quad (5.72)$$

For adiabatic flow, comparison of mass flow rate at point A and point B leads to

$$\begin{aligned} P_0A^*|_A &= P_0A^*|_B \\ \leadsto \frac{P_0|_A}{P_0|_B} &= \frac{A^*|_A}{A^*|_B} \end{aligned} \quad (5.73)$$

And utilizing the equality of  $A^* = \frac{A^*}{A}A$  leads to

$$\frac{P_0|_A}{P_0|_B} = \frac{\frac{A^*}{A}|_{M_A} A|_A}{\frac{A^*}{A}|_{M_B} A|_B} \quad (5.74)$$

For a flow with a constant stagnation pressure (frictionless flow) and non adiabatic flow reads

$$\frac{T_0|_A}{T_0|_B} = \left[ \frac{\frac{B}{A^*}|_{M_B}}{\frac{A}{A^*}|_{M_A}} \frac{A|_B}{A|_A} \right]^2 \quad (5.75)$$

**Example 5.6:**

At point A of the tube the pressure is 3[Bar], Mach number is 2.5, and the duct section area is 0.01[m<sup>2</sup>]. Downstream at exit of tube, point B, the cross section area is 0.015[m<sup>2</sup>] and Mach number is 1.5. Assume no mass lost and adiabatic steady state flow, calculate the total pressure lost.

SOLUTION

Both Mach numbers are known, thus the area ratios can be calculated. The total pressure can be calculated because the Mach number and static pressure are known. With these information, and utilizing equation (5.74) the stagnation pressure at point B can be obtained.

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
1.5000	0.68966	0.39498	1.1762	0.27240	0.32039	0.55401
2.5000	0.44444	0.13169	2.6367	0.05853	0.15432	0.62693

First, the stagnation at point A is obtained from Table (5.2) as

$$P_0|_A = \frac{P}{\underbrace{\left(\frac{P}{P_0}\right)}_{M=2.5}|_A} = \frac{3}{0.058527663} = 51.25781291[\text{Bar}]$$

by utilizing equation (5.74) provides

$$P_0|_B = 51.25781291 \times \frac{1.1761671}{2.6367187} \times \frac{0.01}{0.015} \approx 15.243[\text{Bar}]$$

Hence

$$P_0|_A - P_0|_B = 51.257 - 15.243 = 36.013[\text{Bar}]$$

Note that the large total pressure loss is much larger than the static pressure loss (Pressure point B the pressure is  $0.27240307 \times 15.243 = 4.146[\text{Bar}]$ ).

---

End solution

### 5.3 Isentropic Tables

Table -5.2. Isentropic Table  $k = 1.4$

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.000	1.00000	1.00000	5.8E+5	1.0000	5.8E + 5	2.4E+5
0.050	0.99950	0.99875	11.59	0.99825	11.57	4.838
0.100	0.99800	0.99502	5.822	0.99303	5.781	2.443
0.200	0.99206	0.98028	2.964	0.97250	2.882	1.268
0.300	0.98232	0.95638	2.035	0.93947	1.912	0.89699
0.400	0.96899	0.92427	1.590	0.89561	1.424	0.72632
0.500	0.95238	0.88517	1.340	0.84302	1.130	0.63535
0.600	0.93284	0.84045	1.188	0.78400	0.93155	0.58377
0.700	0.91075	0.79158	1.094	0.72093	0.78896	0.55425
0.800	0.88652	0.73999	1.038	0.65602	0.68110	0.53807
0.900	0.86059	0.68704	1.009	0.59126	0.59650	0.53039
0.95	0.00328	1.061	1.002	1.044	0.95781	1.017
0.96	0.00206	1.049	1.001	1.035	0.96633	1.013
0.97	0.00113	1.036	1.001	1.026	0.97481	1.01
0.98	0.000495	1.024	1	1.017	0.98325	1.007
0.99	0.000121	1.012	1	1.008	0.99165	1.003
1.00	0.83333	0.63394	1.000	0.52828	0.52828	0.52828
1.100	0.80515	0.58170	1.008	0.46835	0.47207	0.52989
1.200	0.77640	0.53114	1.030	0.41238	0.42493	0.53399
1.300	0.74738	0.48290	1.066	0.36091	0.38484	0.53974
1.400	0.71839	0.43742	1.115	0.31424	0.35036	0.54655
1.500	0.68966	0.39498	1.176	0.27240	0.32039	0.55401
1.600	0.66138	0.35573	1.250	0.23527	0.29414	0.56182
1.700	0.63371	0.31969	1.338	0.20259	0.27099	0.56976
1.800	0.60680	0.28682	1.439	0.17404	0.25044	0.57768
1.900	0.58072	0.25699	1.555	0.14924	0.23211	0.58549
2.000	0.55556	0.23005	1.688	0.12780	0.21567	0.59309
2.500	0.44444	0.13169	2.637	0.058528	0.15432	0.62693
3.000	0.35714	0.076226	4.235	0.027224	0.11528	0.65326
3.500	0.28986	0.045233	6.790	0.013111	0.089018	0.67320
4.000	0.23810	0.027662	10.72	0.00659	0.070595	0.68830
4.500	0.19802	0.017449	16.56	0.00346	0.057227	0.69983
5.000	0.16667	0.011340	25.00	0.00189	0.047251	0.70876
5.500	0.14184	0.00758	36.87	0.00107	0.039628	0.71578
6.000	0.12195	0.00519	53.18	0.000633	0.033682	0.72136
6.500	0.10582	0.00364	75.13	0.000385	0.028962	0.72586
7.000	0.092593	0.00261	1.0E+2	0.000242	0.025156	0.72953
7.500	0.081633	0.00190	1.4E+2	0.000155	0.022046	0.73257
8.000	0.072464	0.00141	1.9E+2	0.000102	0.019473	0.73510

Table -5.2. Isentropic Table  $k=1.4$  (continue)

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
8.500	0.064725	0.00107	$2.5E+2$	$6.90E-5$	0.017321	0.73723
9.000	0.058140	0.000815	$3.3E+2$	$4.74E-5$	0.015504	0.73903
9.500	0.052493	0.000631	$4.2E+2$	$3.31E-5$	0.013957	0.74058
10.00	0.047619	0.000495	$5.4E+2$	$2.36E-5$	0.012628	0.74192

(Largest tables in the world can be found in Potto Gas Tables at [www.potto.org](http://www.potto.org))

### 5.3.1 Isentropic Isothermal Flow Nozzle

### 5.3.2 General Relationship

In this section, the other extreme case model where the heat transfer to the gas is perfect, (e.g. Eckert number is very small) is presented. Again in reality the heat transfer is somewhere in between the two extremes. So, knowing the two limits provides a tool to examine where the reality should be expected. The perfect gas model is again assumed (later more complex models can be assumed and constructed in a future versions). In isothermal process the perfect gas model reads

$$P = \rho RT \rightsquigarrow dP = d\rho RT \quad (5.76)$$

Substituting equation (5.76) into the momentum equation<sup>6</sup> yields

$$U dU + \frac{RT dP}{P} = 0 \quad (5.77)$$

Integration of equation (5.77) yields the Bernoulli's equation for ideal gas in isothermal process which reads

$$\rightsquigarrow \frac{U_2^2 - U_1^2}{2} + RT \ln \frac{P_2}{P_1} = 0 \quad (5.78)$$

Thus, the velocity at point 2 becomes

$$U_2 = \sqrt{2RT \ln \frac{P_2}{P_1} - U_1^2} \quad (5.79)$$

The velocity at point 2 for stagnation point,  $U_1 \approx 0$  reads

$$U_2 = \sqrt{2RT \ln \frac{P_2}{P_1}} \quad (5.80)$$

<sup>6</sup>The one dimensional momentum equation for steady state is  $U dU/dx = -dP/dx + 0$  (other effects) which are neglected here.

Or in explicit terms of the stagnation properties the velocity is

$$U = \sqrt{2RT \ln \frac{P}{P_0}} \quad (5.81)$$

Transform from equation (5.78) to a dimensionless form becomes

$$\leadsto \frac{kR\cancel{P} \overset{constant}{(M_2^2 - M_1^2)}}{2} = R\cancel{P} \overset{constant}{\ln \frac{P_2}{P_1}} \quad (5.82)$$

Simplifying equation (5.82) yields

$$\leadsto \frac{k(M_2^2 - M_1^2)}{2} = \ln \frac{P_2}{P_1} \quad (5.83)$$

Or in terms of the pressure ratio equation (5.83) reads

$$\frac{P_2}{P_1} = e^{\frac{k(M_1^2 - M_2^2)}{2}} = \left( \frac{e^{M_1^2}}{e^{M_2^2}} \right)^{\frac{k}{2}} \quad (5.84)$$

As oppose to the adiabatic case ( $T_0 = constant$ ) in the isothermal flow the stagnation temperature ratio can be expressed

$$\frac{T_{01}}{T_{02}} = \frac{T_1 \left( 1 + \frac{k-1}{2} M_1^2 \right)}{T_2 \left( 1 + \frac{k-1}{2} M_2^2 \right)} = \frac{\left( 1 + \frac{k-1}{2} M_1^2 \right)}{\left( 1 + \frac{k-1}{2} M_2^2 \right)} \quad (5.85)$$

Utilizing conservation of the mass  $A\rho M = constant$  to yield

$$\frac{A_1}{A_2} = \frac{M_2 P_2}{M_1 P_1} \quad (5.86)$$

Combining equation (5.86) and equation (5.84) yields

$$\frac{A_2}{A_1} = \frac{M_1}{M_2} \left( \frac{e^{M_2^2}}{e^{M_1^2}} \right)^{\frac{k}{2}} \quad (5.87)$$

The change in the stagnation pressure can be expressed as

$$\frac{P_{02}}{P_{01}} = \frac{P_2}{P_1} \left( \frac{1 + \frac{k-1}{2} M_2^2}{1 + \frac{k-1}{2} M_1^2} \right)^{\frac{k}{k-1}} = \left[ \frac{e^{M_1^2}}{e^{M_2^2}} \right]^{\frac{k}{2}} \quad (5.88)$$

The critical point, at this stage, is unknown (at what Mach number the nozzle is choked is unknown) so there are two possibilities: the choking point or  $M = 1$  to normalize the equation. Here the critical point defined as the point where  $M = 1$  so results can be compared to the adiabatic case and denoted by star. Again it has to emphasis that this critical point is not really related to physical critical point but it is arbitrary definition. The true critical point is when flow is choked and the relationship between two will be presented.

The critical pressure ratio can be obtained from (5.84) to read

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = e^{\frac{(1-M^2)k}{2}} \quad (5.89)$$

Equation (5.87) is reduced to obtained the critical area ratio writes

$$\frac{A}{A^*} = \frac{1}{M} e^{\frac{(1-M^2)k}{2}} \quad (5.90)$$

Similarly the stagnation temperature reads

$$\frac{T_0}{T_0^*} = \frac{2 \left(1 + \frac{k-1}{2} M_1^2\right)^{\frac{k}{k-1}}}{k+1} \quad (5.91)$$

Finally, the critical stagnation pressure reads

$$\frac{P_0}{P_0^*} = e^{\frac{(1-M^2)k}{2}} \left( \frac{2 \left(1 + \frac{k-1}{2} M_1^2\right)^{\frac{k}{k-1}}}{k+1} \right)^{\frac{k}{k-1}} \quad (5.92)$$

The maximum value of stagnation pressure ratio is obtained when  $M = 0$  at which is

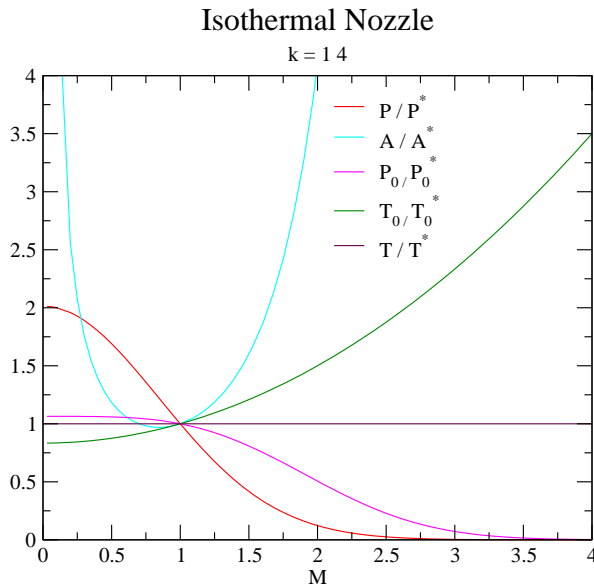
$$\frac{P_0}{P_0^*} \Big|_{M=0} = e^{\frac{k}{2}} \left( \frac{2}{k+1} \right)^{\frac{k}{k-1}} \quad (5.93)$$

For specific heat ratio of  $k = 1.4$ , this maximum value is about two. It can be noted that the stagnation pressure is monotonically reduced during this process.

Of course in isothermal process  $T = T^*$ . All these equations are plotted in Figure (5.6). From the Figure 5.3 it can be observed that minimum of the curve  $A/A^*$  isn't on  $M = 1$ . The minimum of the curve is when area is minimum and at the point where the flow is choked. It should be noted that the stagnation temperature is not constant as in the adiabatic case and the critical point is the only one constant.

The mathematical procedure to find the minimum is simply taking the derivative and equating to zero as following

$$\frac{d \left( \frac{A}{A^*} \right)}{dM} = \frac{kM^2 e^{\frac{k(M^2-1)}{2}} - e^{\frac{k(M^2-1)}{2}}}{M^2} = 0 \quad (5.94)$$



Tue Apr 5 10:20:36 2005

Fig. -5.6. Various ratios as a function of Mach number for isothermal Nozzle

Equation (5.94) simplified to

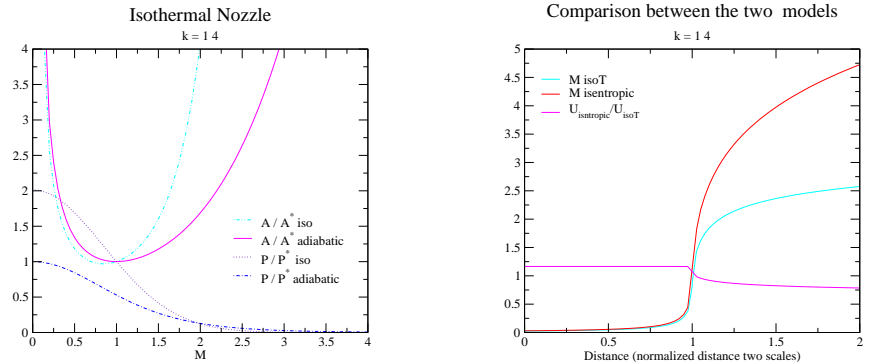
$$kM^2 - 1 = 0 \rightsquigarrow M = \frac{1}{\sqrt{k}} \tag{5.95}$$

It can be noticed that a similar results are obtained for adiabatic flow. The velocity at the throat of isothermal model is smaller by a factor of  $\sqrt{k}$ . Thus, dividing the critical adiabatic velocity by  $\sqrt{k}$  results in

$$U_{throat_{max}} = \sqrt{RT} \tag{5.96}$$

On the other hand, the pressure loss in adiabatic flow is milder as can be seen in Figure (5.7(a)).

It should be emphasized that the stagnation pressure decrees. It is convenient to find expression for the ratio of the initial stagnation pressure (the stagnation pressure before entering the nozzle) to the pressure at the throat. Utilizing equation (5.89) the



Tue Apr 5 10:39:06 2005

(a) Comparison between the isothermal nozzle and adiabatic nozzle in various variables

Thu Apr 7 14:53:49 2005

(b) The comparison of the adiabatic model and isothermal model

Fig. -5.7. The comparison of nozzle flow

following relationship can be obtained

$$\begin{aligned} \frac{P_{throat}}{P_{0_{initial}}} &= \frac{P^*}{P_{0_{initial}}} \frac{P_{throat}}{P^*} = \\ &= \frac{1}{e^{\frac{(1-\theta^2)k}{2}}} e^{\left(1 - \left(\frac{1}{\sqrt{k}}\right)^2\right) \frac{k}{2}} = \\ &= e^{-\frac{1}{2}} = 0.60653 \end{aligned} \quad (5.97)$$

Notice that the critical pressure is independent of the specific heat ratio,  $k$ , as opposed to the adiabatic case. It also has to be emphasized that the stagnation values of the isothermal model are not constant. Again, the heat transfer is expressed as

$$Q = C_p (T_{0_2} - T_{0_1}) \quad (5.98)$$

For comparison between the adiabatic model and the isothermal a simple profile of nozzle area as a function of the distance is assumed. This profile isn't an ideal profile but rather a simple sample just to examine the difference between the two models so in an actual situation it can be bounded. To make sense and eliminate unnecessary details the distance from the entrance to the throat is normalized (to one (1)). In the same fashion the distance from the throat to the exit is normalized (to one (1)) (it doesn't mean that these distances are the same). In this comparison the entrance area ratio and the exit area ratio are the same and equal to 20. The Mach number was computed for the two models and plotted in Figure (5.7(b)). In this comparison it has to be remembered that critical area for the two models are different by about

3% (for  $k = 1.4$ ). As can be observed from Figure (5.7(b)). The Mach number for the isentropic is larger for the supersonic branch but the velocity is lower. The ratio of the velocities can be expressed as

$$\frac{U_s}{U_T} = \frac{M_s \sqrt{kRT_s}}{M_T \sqrt{kRT_s}} \quad (5.99)$$

It can be noticed that temperature in the isothermal model is constant while temperature in the adiabatic model can be expressed as a function of the stagnation temperature. The initial stagnation temperatures are almost the same and can be canceled out to obtain

$$\frac{U_s}{U_T} \sim \frac{M_s}{M_T \sqrt{1 + \frac{k-1}{2} M_s^2}} \quad (5.100)$$

By utilizing equation (5.100) the velocity ratio was obtained and is plotted in Figure (5.7(b)).

Thus, using the isentropic model results in under prediction of the actual results for the velocity in the supersonic branch. While, the isentropic for the subsonic branch will be over prediction. The prediction of the Mach number are similarly shown in Figure (5.7(b)).

Two other ratios need to be examined: temperature and pressure. The initial stagnation temperature is denoted as  $T_{0int}$ . The temperature ratio of  $T/T_{0int}$  can be obtained via the isentropic model as

$$\frac{T}{T_{0int}} = \frac{1}{1 + \frac{k-1}{2} M^2} \quad (5.101)$$

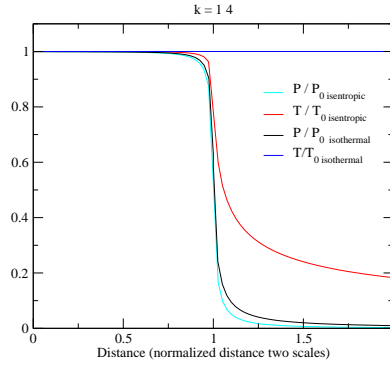
While the temperature ratio of the isothermal model is constant and equal to one (1). The pressure ratio for the isentropic model is

$$\frac{P}{P_{0int}} = \frac{1}{\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k-1}{k}}} \quad (5.102)$$

and for the isothermal process the stagnation pressure varies and has to be taken into account as the following:

$$\frac{P_z}{P_{0int}} = \frac{P_0^*}{P_{0int}} \frac{P_{0z}}{P_0^*} \overbrace{\frac{P_z}{P_{0z}}}^{isentropic} \quad (5.103)$$

Comparison between the two models



Fri Apr 8 15:11:44 2005

Fig. -5.8. Comparison of the pressure and temperature drop as a function of the normalized length (two scales)

where  $z$  is an arbitrary point on the nozzle. Using equations (5.88) and the isentropic relationship, the sought ratio is provided.

Figure (5.8) shows that the range between the predicted temperatures of the two models is very large, while the range between the predicted pressure by the two models is relatively small. The meaning of this analysis is that transferred heat affects the temperature to a larger degree but the effect on the pressure is much less significant.

To demonstrate the relativity of the approach advocated in this book consider the following example.

#### Example 5.7:

*Consider a diverging–converging nozzle made out of wood (low conductive material) with exit area equal entrance area. The throat area ratio to entrance area is 1:4 respectively. The stagnation pressure is 5[Bar] and the stagnation temperature is 27°C. Assume that the back pressure is low enough to have supersonic flow without shock and  $k = 1.4$ . Calculate the velocity at the exit using the adiabatic model. If the nozzle was made from copper (a good heat conductor) a larger heat transfer occurs, should the velocity increase or decrease? What is the maximum possible increase?*

#### SOLUTION

The first part of the question deals with the adiabatic model i.e. the conservation of the stagnation properties. Thus, with known area ratio and known stagnation Potto–GDC provides the following table:

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$
0.14655	0.99572	0.98934	4.0000	0.98511	3.9405
2.9402	0.36644	0.08129	4.0000	0.02979	0.11915

With the known Mach number and temperature at the exit, the velocity can be calculated. The exit temperature is  $0.36644 \times 300 = 109.9K$ . The exit velocity, then, is

$$U = M\sqrt{kRT} = 2.9402\sqrt{1.4 \times 287 \times 109.9} \sim 617.93[m/sec]$$

Even for the isothermal model, the initial stagnation temperature is given as 300K. Using the area ratio in Figure (5.6) or using the Potto–GDC obtains the following table

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$
1.9910	1.4940	0.51183	4.0000	0.12556	0.50225

The exit Mach number is known and the initial temperature to the throat temperature ratio can be calculated as the following:

$$\frac{T_{0_{ini}}}{T_0^*} = \frac{1}{1 + \frac{k-1}{2} \frac{1}{k}} = \frac{1}{1 + \frac{k-1}{k}} = 0.777777778$$

Thus the stagnation temperature at the exit is

$$\frac{T_{0_{ini}}}{T_{0_{exit}}} = 1.4940/0.777777778 = 1.921$$

The exit stagnation temperature is  $1.92 \times 300 = 576.2K$ . The exit velocity can be determined by utilizing the following equation

$$U_{exit} = M\sqrt{kRT} = 1.9910\sqrt{1.4 \times 287 \times 300.0} = 691.253[m/sec]$$

As was discussed before, the velocity in the copper nozzle will be larger than the velocity in the wood nozzle. However, the maximum velocity cannot exceed the  $691.253[m/sec]$

End solution

## 5.4 The Impulse Function

### 5.4.1 Impulse in Isentropic Adiabatic Nozzle

One of the functions that is used in calculating the forces is the Impulse function. The Impulse function is denoted here as  $F$ , but in the literature some denote this function as  $I$ . To explain the motivation for using this definition consider the calculation of the net forces that acting on section shown in Figure (5.9). To calculate the net forces acting in the  $x$ -direction the momentum equation has to be applied

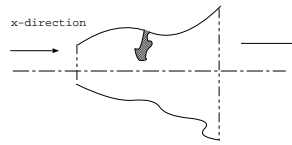


Fig. -5.9. Schematic to explain the significances of the Impulse function.

$$F_{net} = \dot{m}(U_2 - U_1) + P_2A_2 - P_1A_1 \quad (5.104)$$

The net force is denoted here as  $F_{net}$ . The mass conservation also can be applied to our control volume

$$\dot{m} = \rho_1A_1U_1 = \rho_2A_2U_2 \quad (5.105)$$

Combining equation (5.104) with equation (5.105) and by utilizing the identity in equation (5.42) results in

$$F_{net} = kP_2A_2M_2^2 - kP_1A_1M_1^2 + P_2A_2 - P_1A_1 \quad (5.106)$$

Rearranging equation (5.106) and dividing it by  $P_0A^*$  results in

$$\frac{F_{net}}{P_0A^*} = \frac{P_2A_2}{P_0A^*} \overbrace{\left( \frac{f(M_2)}{1 + kM_2^2} \right)} - \frac{P_1A_1}{P_0A^*} \overbrace{\left( \frac{f(M_1)}{1 + kM_1^2} \right)} \quad (5.107)$$

Examining equation (5.107) shows that the right hand side is only a function of Mach number and specific heat ratio,  $k$ . Hence, if the right hand side is only a function of the Mach number and  $k$  than the left hand side must be function of only the same parameters,  $M$  and  $k$ . Defining a function that depends only on the Mach number creates the convenience for calculating the net forces acting on any device. Thus, defining the Impulse function as

$$F = PA(1 + kM_2^2) \quad (5.108)$$

In the Impulse function when  $F$  ( $M = 1$ ) is denoted as  $F^*$

$$F^* = P^*A^*(1 + k) \quad (5.109)$$

The ratio of the Impulse function is defined as

$$\frac{F}{F^*} = \frac{P_1A_1(1 + kM_1^2)}{P^*A^*(1 + k)} = \frac{1}{\underbrace{\frac{P_0}{P^*}}_{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}}} \overbrace{\frac{P_1A_1}{P_0A^*}(1 + kM_1^2)}^{\text{see function (5.107)}} \frac{1}{(1 + k)} \quad (5.110)$$

This ratio is different only in a coefficient from the ratio defined in equation (5.107) which makes the ratio a function of  $k$  and the Mach number. Hence, the net force is

$$F_{net} = P_0A^*(1 + k) \left(\frac{k + 1}{2}\right)^{\frac{k}{k-1}} \left(\frac{F_2}{F^*} - \frac{F_1}{F^*}\right) \quad (5.111)$$

To demonstrate the usefulness of the this function consider a simple situation of the flow through a converging nozzle

#### Example 5.8:

Consider a flow of gas into a converging nozzle with a mass flow rate of  $1[\text{kg}/\text{sec}]$  and the entrance area is  $0.009[\text{m}^2]$  and the exit area is  $0.003[\text{m}^2]$ . The stagnation temperature is  $400\text{K}$  and the pressure at point 2 was measured as  $5[\text{Bar}]$  Calculate the net force acting on the nozzle and pressure at point 1.

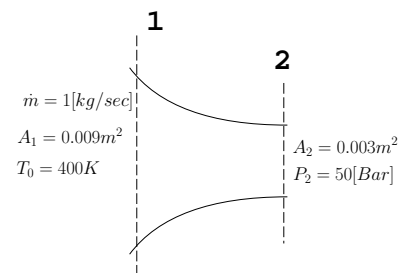


Fig. -5.10. Schematic of a flow of a compressible substance (gas) through a converging nozzle for example (5.8)

#### SOLUTION

The solution is obtained by getting the data for the Mach number. To obtained the

Mach number, the ratio of  $P_1 A_1 / A^* P_0$  is needed to be calculated. To obtain this ratio the denominator is needed to be obtained. Utilizing Fliegner's equation (5.51), provides the following

$$A^* P_0 = \frac{\dot{m} \sqrt{RT}}{0.058} = \frac{1.0 \times \sqrt{400 \times 287}}{0.058} \sim 70061.76 [N]$$

and

$$\frac{A_2 P_2}{A^* P_0} = \frac{500000 \times 0.003}{70061.76} \sim 2.1$$

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.27353	0.98526	0.96355	2.2121	0.94934	2.1000	0.96666

With the area ratio of  $\frac{A}{A^*} = 2.2121$  the area ratio of at point 1 can be calculated.

$$\frac{A_1}{A^*} = \frac{A_2}{A^*} \frac{A_1}{A_2} = 2.2121 \times \frac{0.009}{0.003} = 5.2227$$

And utilizing again Potto-GDC provides

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.11164	0.99751	0.99380	5.2227	0.99132	5.1774	2.1949

The pressure at point 1 is

$$P_1 = P_2 \frac{P_0}{P_2} \frac{P_1}{P_0} = 5.0 \times 0.94934 / 0.99380 \sim 4.776 [Bar]$$

The net force is obtained by utilizing equation (5.111)

$$\begin{aligned} F_{net} &= P_2 A_2 \frac{P_0 A^*}{P_2 A_2} (1+k) \left( \frac{k+1}{2} \right)^{\frac{k}{k-1}} \left( \frac{F_2}{F^*} - \frac{F_1}{F^*} \right) \\ &= 500000 \times \frac{1}{2.1} \times 2.4 \times 1.2^{3.5} \times (2.1949 - 0.96666) \sim 614 [kN] \end{aligned}$$

---

End solution

### 5.4.2 The Impulse Function in Isothermal Nozzle

Previously Impulse function was developed in the isentropic adiabatic flow. The same is done here for the isothermal nozzle flow model. As previously, the definition of the

Impulse function is reused. The ratio of the impulse function for two points on the nozzle is

$$\frac{F_2}{F_1} = \frac{P_2 A_2 + \rho_2 U_2^2 A_2}{P_1 A_1 + \rho_1 U_1^2 A_1} \quad (5.112)$$

Utilizing the ideal gas model for density and some rearrangement results in

$$\frac{F_2}{F_1} = \frac{P_2 A_2}{P_1 A_1} \frac{1 + \frac{U_2^2}{RT}}{1 + \frac{U_1^2}{RT}} \quad (5.113)$$

Since  $U^2/RT = kM^2$  and the ratio of equation (5.86) transformed equation into (5.113)

$$\frac{F_2}{F_1} = \frac{M_2}{M_1} \frac{1 + kM_2^2}{1 + kM_1^2} \quad (5.114)$$

At the star condition ( $M = 1$ ) (not the minimum point) results in

$$\frac{F_2}{F^*} = \frac{M_2}{1} \frac{1 + kM_2^2}{1 + k} \quad (5.115)$$

## 5.5 Isothermal Table

Table -5.3. Isothermal Table

M	$\frac{T_0}{T_0^*}$	$\frac{P_0}{P_0^*}$	$\frac{A}{A^*}$	$\frac{P}{P^*}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.00	0.52828	1.064	5.0E + 5	2.014	1.0E+6	4.2E+5
0.05	0.52921	1.064	9.949	2.010	20.00	8.362
0.1	0.53199	1.064	5.001	2.000	10.00	4.225
0.2	0.54322	1.064	2.553	1.958	5.000	2.200
0.3	0.56232	1.063	1.763	1.891	3.333	1.564
0.4	0.58985	1.062	1.389	1.800	2.500	1.275
0.5	0.62665	1.059	1.183	1.690	2.000	1.125
0.6	0.67383	1.055	1.065	1.565	1.667	1.044
0.7	0.73278	1.047	0.99967	1.429	1.429	1.004
0.8	0.80528	1.036	0.97156	1.287	1.250	0.98750
0.9	0.89348	1.021	0.97274	1.142	1.111	0.98796
1.00	1.000	1.000	1.000	1.000	1.000	1.000
1.10	1.128	0.97376	1.053	0.86329	0.90909	1.020
1.20	1.281	0.94147	1.134	0.73492	0.83333	1.047

Table -5.3. Isothermal Table (continue)

M	$\frac{T_0}{T_0^*}$	$\frac{P_0}{P_0^*}$	$\frac{A}{A^*}$	$\frac{P}{P^*}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
1.30	1.464	0.90302	1.247	0.61693	0.76923	1.079
1.40	1.681	0.85853	1.399	0.51069	0.71429	1.114
1.50	1.939	0.80844	1.599	0.41686	0.66667	1.153
1.60	2.245	0.75344	1.863	0.33554	0.62500	1.194
1.70	2.608	0.69449	2.209	0.26634	0.58824	1.237
1.80	3.035	0.63276	2.665	0.20846	0.55556	1.281
1.90	3.540	0.56954	3.271	0.16090	0.52632	1.328
2.00	4.134	0.50618	4.083	0.12246	0.50000	1.375
2.50	9.026	0.22881	15.78	0.025349	0.40000	1.625
3.000	19.41	0.071758	90.14	0.00370	0.33333	1.889
3.500	40.29	0.015317	$7.5E+2$	0.000380	0.28571	2.161
4.000	80.21	0.00221	$9.1E+3$	$2.75E-5$	0.25000	2.438
4.500	$1.5E+2$	0.000215	$1.6E+5$	$1.41E-6$	0.22222	2.718
5.000	$2.8E+2$	$1.41E-5$	$4.0E+6$	0.0	0.20000	3.000
5.500	$4.9E+2$	0.0	$1.4E+8$	0.0	0.18182	3.284
6.000	$8.3E+2$	0.0	$7.3E+9$	0.0	0.16667	3.569
6.500	$1.4E+3$	0.0	$5.3E+11$	0.0	0.15385	3.856
7.000	$2.2E+3$	0.0	$5.6E+13$	0.0	0.14286	4.143
7.500	$3.4E+3$	0.0	$8.3E+15$	0.0	0.13333	4.431
8.000	$5.2E+3$	0.0	$1.8E+18$	0.0	0.12500	4.719
8.500	$7.7E+3$	0.0	$5.4E+20$	0.0	0.11765	5.007
9.000	$1.1E+4$	0.0	$2.3E+23$	0.0	0.11111	5.296
9.500	$1.6E+4$	0.0	$1.4E+26$	0.0	0.10526	5.586
10.00	$2.2E+4$	0.0	$1.2E+29$	0.0	0.10000	5.875

## 5.6 The effects of Real Gases

To obtain expressions for non-ideal gas it is commonly done by reusing the ideal gas model and introducing a new variable which is a function of the gas properties like the critical pressure and critical temperature. Thus, a real gas equation can be expressed in equation (4.22). Differentiating equation (4.22) and dividing by equation (4.22) yields

$$\frac{dP}{P} = \frac{dz}{z} + \frac{d\rho}{\rho} + \frac{dT}{T} \quad (5.116)$$

Again, Gibb's equation (5.27) is reused to relate the entropy change to the change in thermodynamics properties and applied on non-ideal gas. Since  $ds = 0$  and utilizing the equation of the state  $dh = dP/\rho$ . The enthalpy is a function of the temperature and pressure thus,  $h = h(T, P)$  and full differential is

$$dh = \left( \frac{\partial h}{\partial T} \right)_P dT + \left( \frac{\partial h}{\partial P} \right)_T dP \quad (5.117)$$

The definition of pressure specific heat is  $C_p \equiv \frac{\partial h}{\partial T}$  and second derivative is Maxwell relation hence,

$$\left(\frac{\partial h}{\partial P}\right)_T = v - T \left(\frac{\partial s}{\partial T}\right)_P \quad (5.118)$$

First, the differential of enthalpy is calculated for real gas equation of state as

$$dh = C_p dT - \left(\frac{T}{Z}\right) \left(\frac{\partial z}{\partial T}\right)_P \frac{dP}{\rho} \quad (5.119)$$

Equations (5.27) and (4.22) are combined to form

$$\frac{ds}{R} = \frac{C_p}{R} \frac{dT}{T} - z \left[1 + \left(\frac{T}{Z}\right) \left(\frac{\partial z}{\partial T}\right)_P\right] \frac{dP}{P} \quad (5.120)$$

The mechanical energy equation can be expressed as

$$\int d\left(\frac{U^2}{2}\right) = - \int \frac{dP}{\rho} \quad (5.121)$$

At the stagnation the definition requires that the velocity is zero. To carry the integration of the right hand side the relationship between the pressure and the density has to be defined. The following power relationship is assumed

$$\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{\frac{1}{n}} \quad (5.122)$$

Notice, that for perfect gas the  $n$  is substituted by  $k$ . With integration of equation (5.121) when using relationship which is defined in equation (5.122) results

$$\frac{U^2}{2} = \int_{P_0}^{P_1} \frac{dP}{\rho} = \int_{P_0}^P \frac{1}{\rho_0} \left(\frac{P_0}{P}\right)^{\frac{1}{n}} dP \quad (5.123)$$

Substituting relation for stagnation density (4.22) results

$$\frac{U^2}{2} = \int_{P_0}^P \frac{z_0 R T_0}{P_0} \left(\frac{P_0}{P}\right)^{\frac{1}{n}} dP \quad (5.124)$$

For  $n > 1$  the integration results in

$$U = \sqrt{z_0 R T_0 \frac{2n}{n-1} \left[1 - \left(\frac{P}{P_0}\right)^{\left(\frac{n-1}{n}\right)}\right]} \quad (5.125)$$

For  $n = 1$  the integration becomes

$$U = \sqrt{2z_0 R T_0 \ln \left(\frac{P_0}{P}\right)} \quad (5.126)$$

It must be noted that  $n$  is a function of the critical temperature and critical pressure. The mass flow rate is regardless to equation of state as following

$$\dot{m} = \rho^* A^* U^* \quad (5.127)$$

Where  $\rho^*$  is the density at the throat (assuming the choking condition) and  $A^*$  is the cross area of the throat. Thus, the mass flow rate in our properties

$$\dot{m} = A^* \overbrace{\frac{P_0}{z_0 R T_0} \left(\frac{P}{P_0}\right)^{\frac{1}{n}}}^{\rho^*} \overbrace{\sqrt{z_0 R T_0 \frac{2n}{n-1} \left[1 - \left(\frac{P}{P_0}\right)^{\frac{n-1}{n}}\right]}}^{U^*} \quad (5.128)$$

For the case of  $n = 1$

$$\dot{m} = A^* \overbrace{\frac{P_0}{z_0 R T_0} \left(\frac{P}{P_0}\right)^{\frac{1}{n}}}^{\rho^*} \overbrace{\sqrt{2z_0 R T_0 \ln \left(\frac{P_0}{P}\right)}}^{U^{**}} \quad (5.129)$$

The Mach number can be obtained by utilizing equation (4.37) to defined the Mach number as

$$M = \frac{U}{\sqrt{znRT}} \quad (5.130)$$

Integrating equation (5.120) when  $ds = 0$  results

$$\int_{T_1}^{T_2} \frac{C_p}{R} \frac{dT}{T} = \int_{P_1}^{P_2} z \left(1 + \left(\frac{T}{Z}\right) \left(\frac{\partial z}{\partial T}\right)_P \frac{dP}{P}\right) \quad (5.131)$$

To carryout the integration of equation (5.131) looks at Bernnolli's equation which is

$$\int \frac{dU^2}{2} = - \int \frac{dP}{\rho} \quad (5.132)$$

After integration of the velocity

$$\frac{dU^2}{2} = - \int_1^{P/P_0} \frac{\rho_0}{\rho} d\left(\frac{P}{P_0}\right) \quad (5.133)$$

It was shown in Chapter (4) that (4.36) is applicable for some ranges of relative temperature and pressure (relative to critical temperature and pressure and not the stagnation conditions).

$$U = \sqrt{z_0 R T_0 \left(\frac{2n}{n-1}\right) \left[1 - \left(\frac{P}{P_0}\right)^{\frac{n-1}{n}}\right]} \quad (5.134)$$

When  $n = 1$  or when  $n \rightarrow 1$

$$U = \sqrt{2z_0RT_0 \ln \left( \frac{P_0}{P} \right)} \quad (5.135)$$

The mass flow rate for the real gas  $\dot{m} = \rho^*U^*A^*$

$$\dot{m} = \frac{A^*P_0}{\sqrt{z_0RT_0}} \sqrt{\frac{2n}{n-1}} \left( \frac{P^*}{P_0} \right)^{\frac{1}{n}} \left[ 1 - \frac{P^*}{P_0} \right] \quad (5.136)$$

And for  $n = 1$

$$\dot{m} = \frac{A^*P_0}{\sqrt{z_0RT_0}} \sqrt{\frac{2n}{n-1}} \sqrt{2z_0RT_0 \ln \left( \frac{P_0}{P} \right)} \quad (5.137)$$

Fliegner's number in this case is

$$Fn = \frac{\dot{m}c_0}{A^*P_0} \sqrt{\frac{2n}{n-1}} \left( \frac{P^*}{P_0} \right)^{\frac{1}{n}} \left[ 1 - \frac{P^*}{P_0} \right] \quad (5.138)$$

Fliegner's number for  $n = 1$  is

$$Fn = \frac{\dot{m}c_0}{A^*P_0} = 2 \left( \frac{P^*}{P_0} \right)^2 - \ln \left( \frac{P^*}{P_0} \right) \quad (5.139)$$

The critical ratio of the pressure is

$$\frac{P^*}{P_0} = \left( \frac{2}{n+1} \right)^{\frac{n}{n-1}} \quad (5.140)$$

When  $n = 1$  or more generally when  $n \rightarrow 1$  this is a ratio approach

$$\frac{P^*}{P_0} = \sqrt{e} \quad (5.141)$$

To obtain the relationship between the temperature and pressure, equation (5.131) can be integrated

$$\frac{T_0}{T} = \left( \frac{P_0}{P} \right)^{\frac{R}{c_p} [z + T \left( \frac{\partial z}{\partial T} \right)_P]} \quad (5.142)$$

The power of the pressure ratio is approaching  $\frac{k-1}{k}$  when  $z$  approaches 1. Note that

$$\frac{T_0}{T} = \left( \frac{z_0}{z} \right) \left( \frac{P_0}{P} \right)^{\frac{1-n}{n}} \quad (5.143)$$

The Mach number at every point at the nozzle can be expressed as

$$M = \sqrt{\left(\frac{2}{n-1}\right) \frac{z_0 T_0}{z T} \left[1 - \left(\frac{P-P_0}{P}\right)^{\frac{1-n}{n}}\right]} \quad (5.144)$$

For  $n = 1$  the Mach number is

$$M = \sqrt{2 \frac{z_0 T_0}{z T} \ln \frac{P_0}{P}} \quad (5.145)$$

The pressure ratio at any point can be expressed as a function of the Mach number as

$$\frac{T_0}{T} = \left[1 + \frac{n-1}{2} M^2\right]^{\left(\frac{n-1}{n}\right) \left[z + T \left(\frac{\partial z}{\partial T}\right)_P\right]} \quad (5.146)$$

for  $n = 1$

$$\frac{T_0}{T} = e^{M^2 \left[z + T \left(\frac{\partial z}{\partial T}\right)_P\right]} \quad (5.147)$$

The critical temperature is given by

$$\frac{T^*}{T_0} = \left(\frac{1+n}{2}\right)^{\left(\frac{n}{1-n}\right) \left[z + T \left(\frac{\partial z}{\partial T}\right)_P\right]} \quad (5.148)$$

and for  $n = 1$

$$\frac{T^*}{T_0} = \sqrt{e^{-\left[z + T \left(\frac{\partial z}{\partial T}\right)_P\right]}} \quad (5.149)$$

The mass flow rate as a function of the Mach number is

$$\dot{m} = \frac{P_0 n}{c_0} M \sqrt{\left(1 + \frac{n-1}{2} M^2\right)^{\frac{n+1}{n-1}}} \quad (5.150)$$

For the case of  $n = 1$  the mass flow rate is

$$\dot{m} = \frac{P_0 A^* n}{c_0} \sqrt{e^{M^2}} \sqrt{\left(1 + \frac{n-1}{2} M^2\right)^{\frac{n+1}{n-1}}} \quad (5.151)$$

**Example 5.9:**

A design is required that at a specific point the Mach number should be  $M = 2.61$ , the pressure  $2[\text{Bar}]$ , and temperature  $300\text{K}$ .

- i. Calculate the area ratio between the point and the throat.
- ii. Calculate the stagnation pressure and the stagnation temperature.
- iii. Are the stagnation pressure and temperature at the entrance different from the point? You can assume that  $k = 1.405$ .

SOLUTION

1. The solution is simplified by using Potto-GDC for  $M = 2.61$  the results are

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$
2.6100	0.42027	0.11761	2.9066	0.04943	0.14366

2. The stagnation pressure is obtained from

$$P_0 = \frac{P_0}{P} P = \frac{2.61}{0.04943} P \sim 52.802 [Bar]$$

The stagnation temperature is

$$T_0 = \frac{T_0}{T} T = \frac{300}{0.42027} \sim 713.82 K$$

3. Of course, the stagnation pressure is constant for isentropic flow.

---

End solution