Note:

CHAPTER 10: ISOTHERMAL

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This chapter is part of the textbook:

"Fundamentals of Compressible Flow"

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Compressible Flow	beta		0.4.8.6	✔
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Dynamics	NSY		0.0.0	✖
Fluid Mechanics	beta		0.2.9	✔
Heat Transfer	NSY	Based on Eckert	0.0.0	✖
Mechanics	NSY		0.0.0	✖
Open Channel Flow	NSY		0.0.0	✖
Statics	early alpha	first chapter	0.0.1	✖
Strength of Material	NSY		0.0.0	✖
Thermodynamics	early alpha		0.0.01	✖
Two/Multi phases flow	NSY	Tel-Aviv's notes	0.0.0	✖

NSY = Not Started Yet

CHAPTER 9

Isothermal Flow

In this chapter a model dealing with gas that flows through a long tube is described. This model has a applicability to situations which occur in a relatively long distance and where heat transfer is relatively rapid so that the temperature can be treated, for engineering purposes, as a constant. For example, this model is applicable when a natural gas flows over several hundreds of meters. Such situations are common in large cities in U.S.A. where natural gas is used for heating. It is more predominant (more applicable) in situations where the gas is pumped over a length of kilometers.

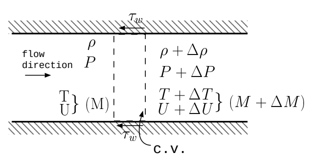

Fig. -9.1. Control volume for isothermal flow.

The high speed of the gas is obtained or explained by the combination of heat transfer and the friction to the flow. For a long pipe, the pressure difference reduces the density of the gas. For instance, in a perfect gas, the density is inverse of the pressure (it has to be kept in mind that the gas undergoes an isothermal process.). To maintain conservation of mass, the velocity increases inversely to the pressure. At critical point the velocity reaches the speed of sound at the exit and hence the flow will be choked[1].

[1]This explanation is not correct as it will be shown later on. Close to the critical point (about, $1/\sqrt{k}$, the heat transfer, is relatively high and the isothermal flow model is not valid anymore. Therefore, the study of the isothermal flow above this point is only an academic discussion but also provides the upper limit for Fanno Flow.

9.1 The Control Volume Analysis/Governing equations

Figure (9.1) describes the flow of gas from the left to the right. The heat transfer up stream (or down stream) is assumed to be negligible. Hence, the energy equation can be written as the following:

$$\frac{dQ}{\dot{m}} = c_p dT + d\frac{U^2}{2} = c_p dT_0 \tag{9.1}$$

The momentum equation is written as the following

$$-AdP - \tau_w dA_{\text{wetted area}} = \dot{m} dU \tag{9.2}$$

where A is the cross section area (it doesn't have to be a perfect circle; a close enough shape is sufficient.). The shear stress is the force per area that acts on the fluid by the tube wall. The $A_{wetted\ area}$ is the area that shear stress acts on. The second law of thermodynamics reads

$$\frac{s_2 - s_1}{C_p} = \ln\frac{T_2}{T_1} - \frac{k-1}{k}\ln\frac{P_2}{P_1} \tag{9.3}$$

The mass conservation is reduced to

$$\dot{m} = constant = \rho U A \tag{9.4}$$

Again it is assumed that the gas is a perfect gas and therefore, equation of state is expressed as the following:

$$P = \rho R T \tag{9.5}$$

9.2 Dimensionless Representation

In this section the equations are transformed into the dimensionless form and presented as such. First it must be recalled that the temperature is constant and therefore, equation of state reads

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{9.6}$$

It is convenient to define a hydraulic diameter

$$D_H = \frac{4 \times \text{Cross Section Area}}{\text{wetted perimeter}} \tag{9.7}$$

Now, the Fanning friction factor[2] is introduced, this factor is a dimensionless friction factor sometimes referred to as the friction coefficient as

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{9.8}$$

Substituting equation (9.8) into momentum equation (9.2) yields

$$-dP - \frac{4dx}{D_H} f \left(\frac{1}{2}\rho U^2\right) = \overbrace{\rho U}^{\frac{\dot{m}}{A}} dU \tag{9.9}$$

Rearranging equation (9.9) and using the identify for perfect gas $M^2 = \rho U^2/kP$ yields:

$$-\frac{dP}{P} - \frac{4f dx}{D_H}\left(\frac{kPM^2}{2}\right) = \frac{kPM^2 dU}{U} \tag{9.10}$$

Now the pressure, P as a function of the Mach number has to substitute along with velocity, U.

$$U^2 = kRTM^2 \tag{9.11}$$

Differentiation of equation (9.11) yields

$$d(U^2) = kR\left(M^2\, dT + T\, d(M^2)\,\right) \tag{9.12}$$

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} - \frac{dT}{T} \tag{9.13}$$

Now it can be noticed that $dT = 0$ for isothermal process and therefore

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} = \frac{2U\, dU}{U^2} = \frac{2dU}{U} \tag{9.14}$$

The dimensionalization of the mass conservation equation yields

$$\frac{d\rho}{\rho} + \frac{dU}{U} = \frac{d\rho}{\rho} + \frac{2UdU}{2U^2} = \frac{d\rho}{\rho} + \frac{d(U^2)}{2\,U^2} = 0 \tag{9.15}$$

Differentiation of the isotropic (stagnation) relationship of the pressure (5.11) yields

$$\frac{dP_0}{P_0} = \frac{dP}{P} + \left(\frac{\frac{1}{2}kM^2}{1 + \frac{k-1}{2}M^2}\right)\frac{dM^2}{M^2} \tag{9.16}$$

[2]It should be noted that Fanning factor based on hydraulic radius, instead of diameter friction equation, thus "Fanning f" values are only 1/4th of "Darcy f" values.

Differentiation of equation (5.9) yields:

$$dT_0 = dT\left(1 + \frac{k-1}{2}M^2\right) + T\,\frac{k-1}{2}\,dM^2 \tag{9.17}$$

Notice that $dT_0 \neq 0$ in an isothermal flow. There is no change in the actual temperature of the flow but the stagnation temperature increases or decreases depending on the Mach number (supersonic flow of subsonic flow). Substituting T for equation (9.17) yields:

$$dT_0 = \frac{T_0\,\frac{k-1}{2}\,d\,M^2}{\left(1 + \frac{k-1}{2}M^2\right)}\,\frac{M^2}{M^2} \tag{9.18}$$

Rearranging equation (9.18) yields

$$\frac{dT_0}{T_0} = \frac{(k-1)\,M^2}{2\left(1 + \frac{k-1}{2}\right)}\,\frac{dM^2}{M^2} \tag{9.19}$$

By utilizing the momentum equation it is possible to obtain a relation between the pressure and density. Recalling that an isothermal flow ($T = 0$) and combining it with perfect gas model yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{9.20}$$

From the continuity equation (see equation (9.14)) leads

$$\frac{dM^2}{M^2} = \frac{2dU}{U} \tag{9.21}$$

The four equations momentum, continuity (mass), energy, state are described above. There are 4 unknowns (M, T, P, ρ)[3] and with these four equations the solution is attainable. One can notice that there are two possible solutions (because of the square power). These different solutions are supersonic and subsonic solution.

The distance friction, $\frac{4fL}{D}$, is selected as the choice for the independent variable. Thus, the equations need to be obtained as a function of $\frac{4fL}{D}$. The density is eliminated from equation (9.15) when combined with equation (9.20) to become

$$\frac{dP}{P} = -\frac{dU}{U} \tag{9.22}$$

[3]Assuming the upstream variables are known.

After substituting the velocity (9.22) into equation (9.10), one can obtain

$$-\frac{dP}{P} - \frac{4fdx}{D_H}\left(\frac{kPM^2}{2}\right) = kPM^2\frac{dP}{P} \tag{9.23}$$

Equation (9.23) can be rearranged into

$$\frac{dP}{P} = \frac{d\rho}{\rho} = -\frac{dU}{U} = -\frac{1}{2}\frac{dM^2}{M^2} = -\frac{kM^2}{2\left(1-kM^2\right)}4f\frac{dx}{D} \tag{9.24}$$

Similarly or by other path the stagnation pressure can be expressed as a function of $\frac{4fL}{D}$

$$\frac{dP_0}{P_0} = \frac{kM^2\left(1-\frac{k+1}{2}M^2\right)}{2\left(kM^2-1\right)\left(1+\frac{k-1}{2}M^2\right)}4f\frac{dx}{D} \tag{9.25}$$

$$\frac{dT_0}{T_0} = \frac{k\left(1-k\right)M^2}{2\left(1-kM^2\right)\left(1+\frac{k-1}{2}M^2\right)}4f\frac{dx}{D} \tag{9.26}$$

The variables in equation (9.24) can be separated to obtain integrable form as follows

$$\int_0^L \frac{4fdx}{D} = \int_{M^2}^{1/k}\frac{1-kM2}{kM2}dM^2 \tag{9.27}$$

It can be noticed that at the entrance ($x = 0$) for which $M = M_{x=0}$ (the initial velocity in the tube isn't zero). The term $\frac{4fL}{D}$ is positive for any x, thus, the term on the other side has to be positive as well. To obtain this restriction $1 = kM^2$. Thus, the value $M = \frac{1}{\sqrt{k}}$ is the limiting case from a mathematical point of view. When Mach number larger than $M > \frac{1}{\sqrt{k}}$ it makes the right hand side of the integrate negative. The physical meaning of this value is similar to $M = 1$ choked flow which was discussed in a variable area flow in Chapter (5).

Further it can be noticed from equation (9.26) that when $M \rightarrow \frac{1}{\sqrt{k}}$ the value of right hand side approaches infinity (∞). Since the stagnation temperature (T_0) has a finite value which means that $dT_0 \rightarrow \infty$. Heat transfer has a limited value therefore the model of the flow must be changed. A more appropriate model is an adiabatic flow model yet it can serve as a bounding boundary (or limit).

Integration of equation (9.27) requires information about the relationship between the length, x, and friction factor f. The friction is a function of the Reynolds number along the tube. Knowing the Reynolds number variations is important. The Reynolds number is defined as

$$Re = \frac{D\,U\,\rho}{\mu} \tag{9.28}$$

The quantity $U\,\rho$ is constant along the tube (mass conservation) under constant area. Thus, the only viscosity is varied along the tube. However under the assumption of ideal gas, viscosity is only a function of the temperature. The temperature in isothermal process (the definition) is constant and thus the viscosity is constant. In real gas the pressure effect is very minimal as described in "Basic of fluid mechanics" by this author. Thus, the friction factor can be integrated to yield

$$\left.\frac{4fL}{D}\right|_{max} = \frac{1-kM^2}{kM^2} + \ln kM^2 \tag{9.29}$$

The definition for perfect gas yields $M^2 = U^2/kRT$ and noticing that $T = constant$ is used to describe the relation of the properties at $M = 1/\sqrt{k}$. By denoting the superscript symbol $*$ for the choking condition, one can obtain that

$$\frac{M^2}{U^2} = \frac{1/k}{{U^*}^2} \tag{9.30}$$

Rearranging equation (9.30) is transformed into

$$\frac{U}{U^*} = \sqrt{k}M \tag{9.31}$$

Utilizing the continuity equation provides

$$\rho U = \rho^* U^*; \Longrightarrow \frac{\rho}{\rho^*} = \frac{1}{\sqrt{k}M} \tag{9.32}$$

Reusing the perfect–gas relationship

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \frac{1}{\sqrt{k}M} \tag{9.33}$$

Now utilizing the relation for stagnated isotropic pressure one can obtain

$$\frac{P_0}{P_0^*} = \frac{P}{P^*}\left[\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2k}}\right]^{\frac{k}{k-1}} \tag{9.34}$$

Substituting for $\frac{P}{P^*}$ equation (9.33) and rearranging yields

$$\frac{P_0}{P_0^*} = \frac{1}{\sqrt{k}}\left(\frac{2k}{3k-1}\right)^{\frac{k}{k-1}}\left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}\frac{1}{M} \tag{9.35}$$

And the stagnation temperature at the critical point can be expressed as

$$\frac{T_0}{T_0^*} = \frac{T}{T^*}\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2k}} = \frac{2k}{3k-1}\left(1+\frac{k-1}{2}\right)M^2 \tag{9.36}$$

These equations (9.31)-(9.36) are presented on in Figure (9.2)

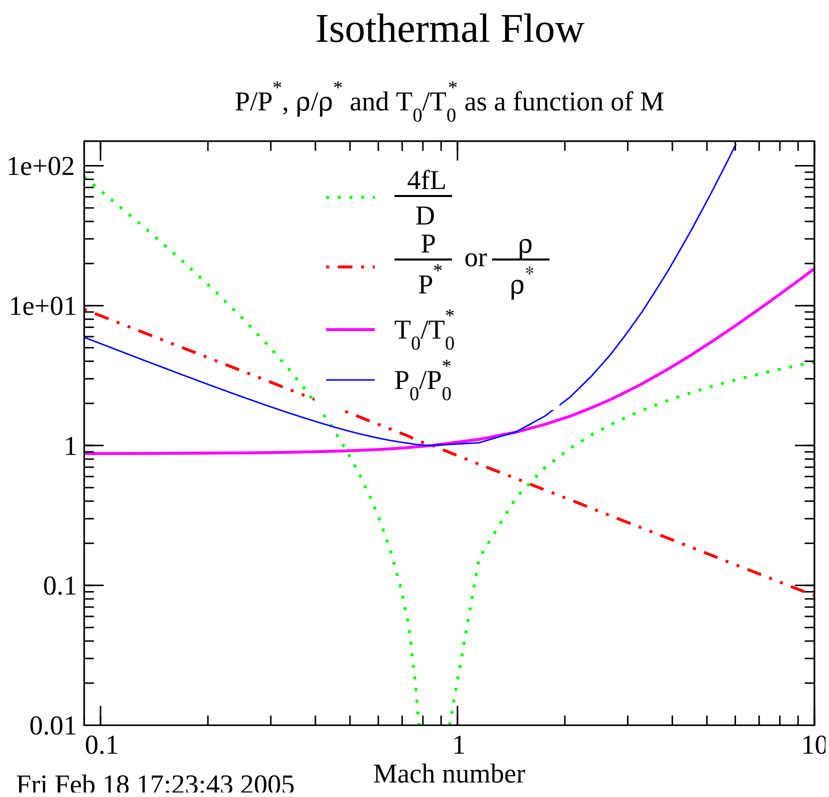

Fig. -9.2. Description of the pressure, temperature relationships as a function of the Mach number for isothermal flow

9.3 The Entrance Limitation of Supersonic Branch

Situations where the conditions at the tube exit have not arrived at the critical conditions are discussed here. It is very useful to obtain the relationship between the entrance and the exit condition for this case. Denote 1 and 2 as the conditions at the inlet and exit respectably. From equation (9.24)

$$\frac{4fL}{D} = \left.\frac{4fL}{D}\right|_{max_1} - \left.\frac{4fL}{D}\right|_{max_2} = \frac{1 - k{M_1}^2}{k{M_1}^2} - \frac{1 - k{M_2}^2}{k{M_2}^2} + \ln\left(\frac{M_1}{M_2}\right)^2 \tag{9.37}$$

For the case that $M_1 >> M_2$ and $M_1 \rightarrow 1$ equation (9.37) is reduced into the following approximation

$$\frac{4fL}{D} = 2\ln M_1 - 1 - \overbrace{\frac{1 - k{M_2}^2}{k{M_2}^2}}^{\sim 0} \tag{9.38}$$

Solving for M_1 results in

$$M_1 \sim \mathbf{e}^{\frac{1}{2}\left(\frac{4fL}{D}+1\right)} \tag{9.39}$$

This relationship shows the maximum limit that Mach number can approach when the heat transfer is extraordinarily fast. In reality, even small $\frac{4fL}{D} > 2$ results in a Mach number which is larger than 4.5. This velocity requires a large entrance length to achieve good heat transfer. With this conflicting mechanism obviously the flow is closer to the Fanno flow model. Yet this model provides the directions of the heat transfer effects on the flow.

9.4 Comparison with Incompressible Flow

The Mach number of the flow in some instances is relatively small. In these cases, one should expect that the isothermal flow should have similar characteristics as incompressible flow. For incompressible flow, the pressure loss is expressed as follows

$$P_1 - P_2 = \frac{4fL}{D}\frac{U^2}{2} \tag{9.40}$$

Now note that for incompressible flow $U_1 = U_2 = U$ and $\frac{4fL}{D}$ represent the ratio of the traditional h_{12}. To obtain a similar expression for isothermal flow, a relationship between M_2 and M_1 and pressures has to be derived. From equation (9.40) one can obtained that

$$M_2 = M_1 \frac{P_1}{P_2} \tag{9.41}$$

Substituting this expression into (9.41) yields

$$\frac{4fL}{D} = \frac{1}{k{M_1}^2}\left(1 - \left(\frac{P_2}{P_1}\right)^2\right) - \ln\left(\frac{P_2}{P_1}\right)^2 \tag{9.42}$$

Because f is always positive there is only one solution to the above equation even though M2.

Expanding the solution for small pressure ratio drop, $P_1 - P_2/P_1$, by some mathematics.

denote

$$\chi = \frac{P_1 - P_2}{P_1} \tag{9.43}$$

Now equation (9.42) can be transformed into

$$\frac{4fL}{D} = \frac{1}{k{M_1}^2}\left(1 - \left(\frac{P_2 - P_1 + P_1}{P_1}\right)^2\right) - \ln\left(\frac{1}{\frac{P_2}{P_1}}\right)^2 \tag{9.44}$$

$$\frac{4fL}{D} = \frac{1}{kM_1{}^2}\left(1-(1-\chi)^2\right) - \ln\left(\frac{1}{1-\chi}\right)^2 \tag{9.45}$$

$$\frac{4fL}{D} = \frac{1}{kM_1{}^2}\left(2\chi-\chi^2\right) - \ln\left(\frac{1}{1-\chi}\right)^2 \tag{9.46}$$

now we have to expand into a series around $\chi = 0$ and remember that

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + 0\left(x^3\right) \tag{9.47}$$

and for example the first derivative of

$$\begin{aligned} \frac{d}{d\chi}\ln\left(\frac{1}{1-\chi}\right)^2\bigg|_{\chi=0} &= \\ (1-\chi)^2 \times \left[(-2)(1-\chi)^{-3}\right](-1)\Big|_{\chi=0} &= \quad 2 \end{aligned} \tag{9.48}$$

similarly it can be shown that $f''(\chi=0) = 1$ equation (9.46) now can be approximated as

$$\frac{4fL}{D} = \frac{1}{kM_1{}^2}(2\chi-\chi^2) - \left(2\chi-\chi^2\right) + f\left(\chi^3\right) \tag{9.49}$$

rearranging equation (9.49) yields

$$\frac{4fL}{D} = \frac{\chi}{kM_1{}^2}\left[(2-\chi) - kM_1{}^2\,(2-\chi)\right] + f\left(\chi^3\right) \tag{9.50}$$

and further rearrangement yields

$$\frac{4fL}{D} = \frac{\chi}{kM_1{}^2}\left[2(1-kM_1{}^2) - \left(1+kM_1{}^2\right)\chi\right] + f\left(\chi^3\right) \tag{9.51}$$

in cases that χ is small

$$\frac{4fL}{D} \approx \frac{\chi}{kM_1{}^2}\left[2(1-kM_1{}^2) - \left(1+kM_1{}^2\right)\chi\right] \tag{9.52}$$

The pressure difference can be plotted as a function of the M_1 for given value of $\frac{4fL}{D}$. Equation (9.52) can be solved explicitly to produce a solution for

$$\chi = \frac{1-kM_1{}^2}{1+kM_1{}^2} - \sqrt{\frac{1-kM_1{}^2}{1+kM_1{}^2} - \frac{kM_1{}^2}{1+kM_1{}^2}\frac{4fL}{D}} \tag{9.53}$$

A few observations can be made about equation (9.53).

9.5 Supersonic Branch

Apparently, this analysis/model is over simplified for the supersonic branch and does not produce reasonable results since it neglects to take into account the heat transfer effects. A dimensionless analysis[4] demonstrates that all the common materials that the author is familiar which creates a large error in the fundamental assumption of the model and the model breaks. Nevertheless, this model can provide a better understanding to the trends and deviations of the Fanno flow model.

In the supersonic flow, the hydraulic entry length is very large as will be shown below. However, the feeding diverging nozzle somewhat reduces the required entry length (as opposed to converging feeding). The thermal entry length is in the order of the hydrodynamic entry length (look at the Prandtl number, (0.7-1.0), value for the common gases.). Most of the heat transfer is hampered in the sublayer thus the core assumption of isothermal flow (not enough heat transfer so the temperature isn't constant) breaks down[5].

The flow speed at the entrance is very large, over hundred of meters per second. For example, a gas flows in a tube with $\frac{4fL}{D} = 10$ the required entry Mach number is over 200. Almost all the perfect gas model substances dealt with in this book, the speed of sound is a function of temperature. For this illustration, for most gas cases the speed of sound is about $300[m/sec]$. For example, even with low temperature like $200K$ the speed of sound of air is $283[m/sec]$. So, even for relatively small tubes with $\frac{4fL}{D} = 10$ the inlet speed is over 56 [km/sec]. This requires that the entrance length to be larger than the actual length of the tub for air. Remember from Fluid Dynamic book

$$L_{entrance} = 0.06 \frac{UD}{\nu} \tag{9.54}$$

The typical values of the the kinetic viscosity, ν, are 0.0000185 kg/m-sec at 300K and 0.0000130034 kg/m-sec at 200K. Combine this information with our case of $\frac{4fL}{D} = 10$

$$\frac{L_{entrance}}{D} = 250746268.7$$

On the other hand a typical value of friction coefficient $f = 0.005$ results in

$$\frac{L_{max}}{D} = \frac{10}{4 \times 0.005} = 500$$

The fact that the actual tube length is only less than 1% of the entry length means that the assumption is that the isothermal flow also breaks (as in a large response time).

Now, if Mach number is changing from 10 to 1 the kinetic energy change is about $\frac{\mathbf{T_0}}{\mathbf{T_0}^*} = 18.37$ which means that the maximum amount of energy is insufficient.

Now with limitation, this topic will be covered in the next version because it provide some insight and boundary to the Fanno Flow model.

[4]This dimensional analysis is a bit tricky, and is based on estimates. Currently and ashamedly the author is looking for a more simplified explanation. The current explanation is correct but based on hands waving and definitely does not satisfy the author.

[5]see Kays and Crawford "Convective Heat Transfer" (equation 12-12).

9.6 Figures and Tables

Table -9.1. The Isothermal Flow basic parameters

$\mathbf{M}$	$\frac{4fL}{D}$	$\frac{P}{P^*}$	$\frac{P_0}{{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{T_0}{{T_0}^*}$
0.03000	785.97	28.1718	17.6651	28.1718	0.87516
0.04000	439.33	21.1289	13.2553	21.1289	0.87528
0.05000	279.06	16.9031	10.6109	16.9031	0.87544
0.06000	192.12	14.0859	8.8493	14.0859	0.87563
0.07000	139.79	12.0736	7.5920	12.0736	0.87586
0.08000	105.89	10.5644	6.6500	10.5644	0.87612
0.09000	82.7040	9.3906	5.9181	9.3906	0.87642
0.10000	66.1599	8.4515	5.3334	8.4515	0.87675
0.20000	13.9747	4.2258	2.7230	4.2258	0.88200
0.25000	7.9925	3.3806	2.2126	3.3806	0.88594
0.30000	4.8650	2.8172	1.8791	2.8172	0.89075
0.35000	3.0677	2.4147	1.6470	2.4147	0.89644
0.40000	1.9682	2.1129	1.4784	2.1129	0.90300
0.45000	1.2668	1.8781	1.3524	1.8781	0.91044
0.50000	0.80732	1.6903	1.2565	1.6903	0.91875
0.55000	0.50207	1.5366	1.1827	1.5366	0.92794
0.60000	0.29895	1.4086	1.1259	1.4086	0.93800
0.65000	0.16552	1.3002	1.0823	1.3002	0.94894
0.70000	0.08085	1.2074	1.0495	1.2074	0.96075
0.75000	0.03095	1.1269	1.0255	1.1269	0.97344
0.80000	0.00626	1.056	1.009	1.056	0.98700
0.81000	0.00371	1.043	1.007	1.043	0.98982
0.81879	0.00205	1.032	1.005	1.032	0.99232
0.82758	0.000896	1.021	1.003	1.021	0.99485
0.83637	0.000220	1.011	1.001	1.011	0.99741
0.84515	0.0	1.000	1.000	1.000	1.000

9.7 Isothermal Flow Examples

There can be several kinds of questions aside from the proof questions[6] Generally, the "engineering" or practical questions can be divided into driving force (pressure difference), resistance (diameter, friction factor, friction coefficient, etc.), and mass flow rate questions. In this model no questions about shock (should) exist[7].

[6]The proof questions are questions that ask for proof or for finding a mathematical identity (normally good for mathematicians and study of perturbation methods). These questions or examples will appear in the later versions.

[7]Those who are mathematically inclined can include these kinds of questions but there are no real world applications to isothermal model with shock.

The driving force questions deal with what should be the pressure difference to obtain certain flow rate. Here is an example.

Example 9.1:
A tube of 0.25 [m] diameter and 5000 [m] in length is attached to a pump. What should be the pump pressure so that a flow rate of 2 $[kg/sec]$ will be achieved? Assume that friction factor $f = 0.005$ and the exit pressure is $1[bar]$. The specific heat for the gas, $k = 1.31$, surroundings temperature 27°C, $R = 290 \left[\frac{J}{Kkg}\right]$. Hint: calculate the maximum flow rate and then check if this request is reasonable.

SOLUTION

If the flow was incompressible then for known density, ρ, the velocity can be calculated by utilizing $\Delta P = \frac{4fL}{D}\frac{U^2}{2g}$. In incompressible flow, the density is a function of the entrance Mach number. The exit Mach number is not necessarily $1/\sqrt{k}$ i.e. the flow is not choked. First, check whether flow is choked (or even possible).

Calculating the resistance, $\frac{4fL}{D}$

$$\frac{4fL}{D} = \frac{4 \times 0.0055000}{0.25} = 400$$

Utilizing Table (9.1) or the program provides

M	$\frac{4fL}{D}$	$\frac{P}{P^*}$	$\frac{P_0}{P_0{}^*}$	$\frac{\rho}{\rho^*}$	$\frac{T_0}{T_0{}^*}$
0.04331	400.00	20.1743	12.5921	0.0	0.89446

The maximum flow rate (the limiting case) can be calculated by utilizing the above table. The velocity of the gas at the entrance $U = cM = 0.04331 \times \sqrt{1.31 \times 290 \times 300} \cong 14.62 \left[\frac{m}{sec}\right]$. The density reads

$$\rho = \frac{P}{RT} = \frac{2,017,450}{290 \times 300} \cong 23.19 \left[\frac{kg}{m^3}\right]$$

The maximum flow rate then reads

$$\dot{m} = \rho AU = 23.19 \times \frac{\pi \times (0.25)^2}{4} \times 14.62 \cong 16.9 \left[\frac{kg}{sec}\right]$$

The maximum flow rate is larger then the requested mass rate hence the flow is not choked. It is note worthy to mention that since the isothermal model breaks around the choking point, the flow rate is really some what different. It is more appropriate to assume an isothermal model hence our model is appropriate.

To solve this problem the flow rate has to be calculated as

$$\dot{m} = \rho AU = 2.0 \left[\frac{kg}{sec}\right]$$

$$\dot{m} = \frac{P_1}{RT} A \frac{kU}{k} = \frac{P_1}{\sqrt{kRT}} A \frac{kU}{\sqrt{kRT}} = \frac{P_1}{c} A k M_1$$

Now combining with equation (9.41) yields

$$\dot{m} = \frac{M_2 P_2 A k}{c}$$

$$M_2 = \frac{\dot{m} c}{P_2 A k} = \frac{2 \times 337.59}{100000 \times \frac{\pi \times (0.25)^2}{4} \times 1.31} = 0.103$$

From Table (9.1) or by utilizing the program

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.10300	66.6779	8.4826	5.3249	0.0	0.89567

The entrance Mach number is obtained by

$$\left.\frac{4fL}{D}\right|_1 = 66.6779 + 400 \cong 466.68$$

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.04014	466.68	21.7678	13.5844	0.0	0.89442

The pressure should be

$$P = 21.76780 \times 8.4826 = 2.566[bar]$$

Note that tables in this example are for $k = 1.31$

End Solution

Example 9.2:

A flow of gas was considered for a distance of 0.5 [km] (500 [m]). A flow rate of 0.2 [kg/sec] is required. Due to safety concerns, the maximum pressure allowed for the gas is only 10[bar]. Assume that the flow is isothermal and k=1.4, calculate the required diameter of tube. The friction coefficient for the tube can be assumed as 0.02 (A relative smooth tube of cast iron.). Note that tubes are provided in increments of 0.5 [in][8]*. You can assume that the soundings temperature to be* 27°C.

SOLUTION

At first, the minimum diameter will be obtained when the flow is choked. Thus, the

[8]It is unfortunate, but it seems that this standard will be around in USA for some time.

maximum M_1 that can be obtained when the M_2 is at its maximum and back pressure is at the atmospheric pressure.

$$M_1 = M_2 \frac{P_2}{P_1} = \overbrace{\frac{1}{\sqrt{k}}}^{M_{max}} \frac{1}{10} = 0.0845$$

Now, with the value of M_1 either by utilizing Table (9.1) or using the provided program yields

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{P}{P^*}}$	$\mathbf{\frac{P_0}{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\mathbf{\frac{T_0}{T_0{}^*}}$
0.08450	94.4310	10.0018	6.2991	0.0	0.87625

With $\left.\frac{4fL}{D}\right|_{max} = 94.431$ the value of minimum diameter.

$$D = \frac{4fL}{\left.\frac{4fL}{D}\right|_{max}} \simeq \frac{4 \times 0.02 \times 500}{94.43} \simeq 0.42359[m] = 16.68[in]$$

However, the pipes are provided only in 0.5 increments and the next size is $17[in]$ or $0.4318[m]$. With this pipe size the calculations are to be repeated in reverse and produces: (Clearly the maximum mass is determined with)

$$\dot{m} = \rho A U = \rho A M c = \frac{P}{RT} A M \sqrt{kRT} = \frac{PAM\sqrt{k}}{\sqrt{RT}}$$

The usage of the above equation clearly applied to the whole pipe. The only point that must be emphasized is that all properties (like Mach number, pressure and etc) have to be taken at the same point. The new $\frac{4fL}{D}$ is

$$\frac{4fL}{D} = \frac{4 \times 0.02 \times 500}{0.4318} \simeq 92.64$$

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{P}{P^*}}$	$\mathbf{\frac{P_0}{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\mathbf{\frac{T_0}{T_0{}^*}}$
0.08527	92.6400	9.9110	6.2424	0.0	0.87627

To check whether the flow rate satisfies the requirement

$$\dot{m} = \frac{10^6 \times \frac{\pi \times 0.4318^2}{4} \times 0.0853 \times \sqrt{1.4}}{\sqrt{287 \times 300}} \approx 50.3[kg/sec]$$

Since $50.3 \geq 0.2$ the mass flow rate requirement is satisfied.

It should be noted that P should be replaced by P_0 in the calculations. The speed of sound at the entrance is

$$c = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 300} \cong 347.2 \left[\frac{m}{sec}\right]$$

and the density is

$$\rho = \frac{P}{RT} = \frac{1,000,000}{287 \times 300} = 11.61 \left[\frac{kg}{m^3}\right]$$

The velocity at the entrance should be

$$U = M * c = 0.08528 \times 347.2 \cong 29.6 \left[\frac{m}{sec}\right]$$

The diameter should be

$$D = \sqrt{\frac{4\dot{m}}{\pi U \rho}} = \sqrt{\frac{4 \times 0.2}{\pi \times 29.6 \times 11.61}} \cong 0.027$$

Nevertheless, for the sake of the exercise the other parameters will be calculated. This situation is reversed question. The flow rate is given with the diameter of the pipe. It should be noted that the flow isn't choked.

End Solution

Example 9.3:

A gas flows of from a station (a) with pressure of 20[bar] through a pipe with 0.4[m] diameter and 4000 [m] length to a different station (b). The pressure at the exit (station (b)) is 2[bar]. The gas and the sounding temperature can be assumed to be 300 K. Assume that the flow is isothermal, k=1.4, and the average friction f=0.01. Calculate the Mach number at the entrance to pipe and the flow rate.

SOLUTION

First, the information whether the flow is choked needs to be found. Therefore, at first it will be assumed that the whole length is the maximum length.

$$\left.\frac{4fL}{D}\right|_{max} = \frac{4 \times 0.01 \times 4000}{0.4} = 400$$

with $\left.\frac{4fL}{D}\right|_{max} = 400$ the following can be written

$\mathbf{M}$	$\mathbf{\frac{4fL}{D}}$	$\mathbf{\frac{T_0}{{T_0}^{*T}}}$	$\frac{\rho}{\rho^{*T}}$	$\mathbf{\frac{P}{P^{*T}}}$	$\mathbf{\frac{P_0}{{P_0}^{*T}}}$
0.0419	400.72021	0.87531	20.19235	20.19235	12.66915

From the table $M_1 \approx 0.0419$,and $\frac{P_0}{{P_0}^{*T}} \approx 12.67$

$${P_0}^{*T} \cong \frac{28}{12.67} \simeq 2.21[bar]$$

The pressure at point (b) by utilizing the isentropic relationship ($M = 1$) pressure ratio is 0.52828.

$$P_2 = \frac{P_0{}^{*T}}{\left(\frac{P_2}{P_0{}^{*T}}\right)} = 2.21 \times 0.52828 = 1.17[bar]$$

As the pressure at point (b) is smaller than the actual pressure $P^* < P_2$ than the actual pressure one must conclude that the flow is not choked. The solution is an iterative process.

1. guess reasonable value of M_1 and calculate $\frac{4fL}{D}$
2. Calculate the value of $\left.\frac{4fL}{D}\right|_2$ by subtracting $\left.\frac{4fL}{D}\right|_1 - \frac{4fL}{D}$
3. Obtain M_2 from the Table ? or by using the Potto–GDC.
4. Calculate the pressure, P_2 bear in mind that this isn't the real pressure but based on the assumption.
5. Compare the results of guessed pressure P_2 with the actual pressure and choose new Mach number M_1 accordingly.

Now the process has been done for you and is provided in figure ??? or in the table obtained from the provided program.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{max}}\right\|_{\mathbf{1}}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.0419	0.59338	400.32131	400.00000	0.10000

The flow rate is

$$\dot{m} = \rho A M c = \frac{P\sqrt{k}}{\sqrt{RT}} \frac{\pi \times D^2}{4} M = \frac{2000000\sqrt{1.4}}{\sqrt{300 \times 287}} \pi \times 0.2^2 \times 0.0419$$
$$\simeq 42.46[kg/sec]$$

End Solution

In this chapter, there are no examples on isothermal with supersonic flow.

9.8 Unchoked situations in Fanno Flow

Table -9.2. The flow parameters for unchoked flow

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{max}}\right\|_{\mathbf{1}}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.7272	0.84095	0.05005	0.05000	0.10000
0.6934	0.83997	0.08978	0.08971	0.10000

Table -9.2. The flow parameters for unchoked flow (continue)

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{max}}\Big\|_{\mathbf{1}}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_2}}$
0.6684	0.84018	0.12949	0.12942	0.10000
0.6483	0.83920	0.16922	0.16912	0.10000
0.5914	0.83889	0.32807	0.32795	0.10000
0.5807	0.83827	0.36780	0.36766	0.10000
0.5708	0.83740	0.40754	0.40737	0.10000

One of the interesting feature of the isothermal flow is that Reynolds number remains constant during the flow for an ideal gas material (enthalpy is a function of only the temperature). This fact simplifies the calculation of the friction factor. This topic has more discussion on the web than on "scientific" literature. Here is a theoretical example for such calculation that was discussed on the web.

Example 9.4:

Air flows in a tube with 0.1$[m]$ *diameter and 100*$[m]$ *in length. The relative roughness,* $\epsilon/D = 0.001$ *and the entrance pressure is* $P_1 = 20[Bar]$ *and the exit pressure is* $P_1 = 1[Bar]$ *. The surroundings temperature is* $27°\mathrm{C}$*. Estimate whether the flow is laminar or turbulent, estimate the friction factor, the entrance and exit Mach numbers and the flow rate.*

<u>SOLUTION</u>

The first complication is the know what is flow regimes. The process is to assume that the flow is turbulent (long pipe). In this case, for large Reynolds number the friction factor is about 0.005. Now the iterative procedure as following;

Calculate the $\frac{4fL}{D}$.

$$\frac{4fL}{D} = \frac{4 \times 0.005 \times 100}{0.1} = 20$$

For this value and the given pressure ratio the flow is choked. Thus,

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.17185	20.0000	4.9179	3.1460	4.9179	0.88017

For this iteration the viscosity of the air is taken from the Basics of Fluid Mechanics by this author and the Reynolds number can be calculated as

$$Re = \frac{D\,U\,\rho}{\mu} = \frac{0.1 \times 0.17185 \times \sqrt{1.4 \times 287 \times 300} \times \dfrac{200000}{287 \times 300}}{0.0008} \sim 17159.15$$

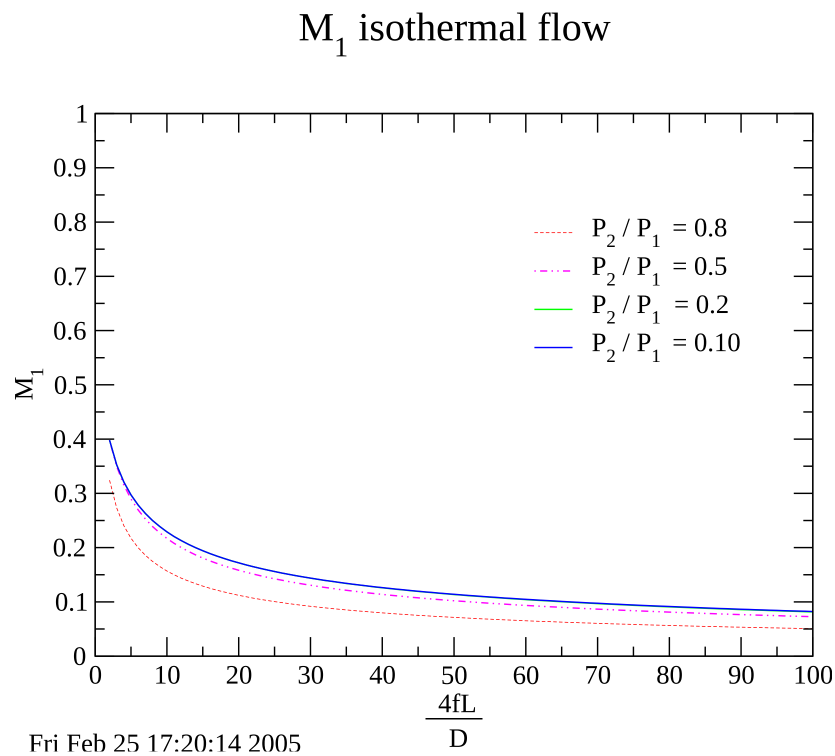

Fig. -9.3. The Mach number at the entrance to a tube under isothermal flow model as a function $\frac{4fL}{D}$

For this Reynolds number the fiction factor can be estimated by using the full Colebrook's equation

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D_h}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) \tag{9.55}$$

or the approximated Haaland's equation

$$\frac{1}{\sqrt{f}} = -1.8\log_{10}\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{Re}\right] \tag{9.56}$$

which provide $f = 0.0053$ and it is a reasonable answer in one iteration. Repeating the iteration results in

$$\frac{4fL}{D} = \frac{4 \times 0.0053 \times 100}{0.1} = 21.2$$

with

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.16689	21.4000	5.0640	3.2357	5.0640	0.87987

And the "improved" Reynolds number is

$$Re = \frac{0.1 \times 0.16689 \times \sqrt{1.4 \times 287 \times 300} \times \dfrac{200000}{287 \times 300}}{0.0008} \sim 16669.6$$

And the friction number is .0054 which very good estimate compare with the assumption that this model was built on.

End Solution