

Note:

CHAPTER 11: TANK

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**“Fundamentals of Compressible
Flow Mechanics”**

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OCTOBER 23, 2009

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Heat Transfer	NSY	Based on Eckert	0.0.0	✗	-
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Open Channel Flow	NSY		0.0.0	✗	-
Statics	early alpha	first chapter	0.0.1	✗	-
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NSY = Not Started Yet

CHAPTER 12

Evacuating and Filling a Semi Rigid Chambers

In some ways the next two Chapters contain materials is new to the traditional compressible flow text books¹. It was the undersigned experience, that in traditional classes for with compressible flow (sometimes referred to as gas dynamics) don't provide a demonstration to applicability of the class material aside to aeronautical spectrum even such as turbomachinery. In this Chapter a discussion on application of compressible flow to other fields like manufacturing is presented².

There is a significant importance to the "pure" models such Isothermal flow and Fanno flow which have immediate applicability. However, in many instances, the situations, in life, are far more complicate. Combination of gas compressibility in the chamber and flow out or through a

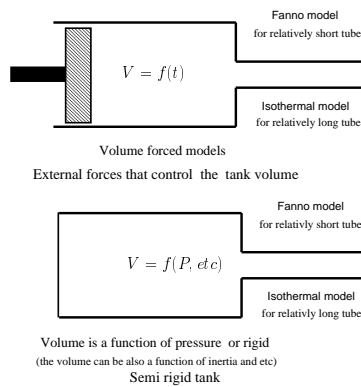


Fig. -12.1. The two different classifications of models that explain the filling or evacuating of a single chamber

¹After completion of these Chapters, the undersigned discover two text books which to include some material related to this topic. These books are OCR, J. A., Fundamentals of Gas Dynamics, International Textbook Co., Scranton, Pennsylvania, 1964. and "Compressible Fluid Flow," 2nd Edition, by M. A. Saad, Prentice Hall, 1985. However, these books contained only limit discussions on the evacuation of chamber with attached nozzle.

²Even if the instructor feels that their students are convinced about the importance of the compressible, this example can further strength and enhance this conviction.

tube post a special interest and these next two Chapters are dealing with these topics. In the first Chapter models, where the chamber volume is controlled or a function of the pressure, are discussed. In the second Chapter, models, where the chamber's volume is a function of external forces, are presented (see Figure (12.1)).

12.1 Governing Equations and Assumptions

The process of filling or evacuating a semi flexible (semi rigid) chamber through a tube is very common in engineering. For example, most car today equipped with an airbag. For instance, the models in this Chapter are suitable for study of the filling the airbag or filling bicycle with air. The analysis is extended to include a semi rigid tank. The term semi rigid tank referred to a tank that the volume is either completely rigid or is a function of the chamber's pressure.

As it was shown in this book the most appropriate model for the flow in the tube for a relatively fast situation is Fanno Flow. The Isothermal model is more appropriate for cases where the tube is relatively long in-which a significant heat transfer occurs keeping the temperature almost constant. As it was shown in Chapter (10) the resistance, $\frac{4fL}{D}$, should be larger than 400. Yet Isothermal flow model is used as the limiting case.

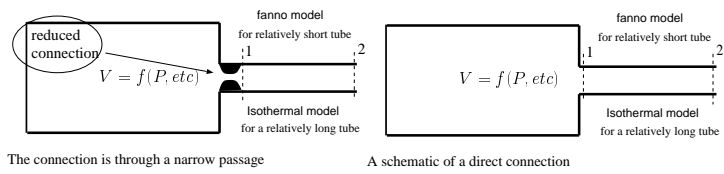


Fig. -12.2. A schematic of two possible connections of the tube to a single chamber

The Rayleigh flow model requires that a constant heat transfer supplied either by chemical reactions or otherwise. This author isn't familiar with situations in which Rayleigh flow model is applicable. And therefore, at this stage, no discussion is offered here.

Fanno flow model is the most appropriate in the case where the filling and evacuating is relatively fast. In case the filling is relatively slow (long $\frac{4fL}{D}$ than the Isothermal flow is appropriate model. Yet as it was stated before, here Isothermal flow and Fanno flow are used as limiting or bounding cases for the real flow. Additionally, the process in the chamber can be limited or bounded between two limits of Isentropic process or Isothermal process.

In this analysis, in order to obtain the essence of the process, some simplified

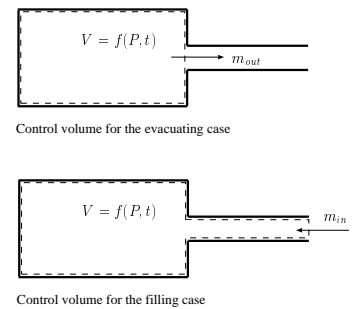


Fig. -12.3. A schematic of the control volumes used in this model

assumptions are made. The assumptions can be relaxed or removed and the model will be more general. Of course, the payment is by far more complex model that sometime clutter the physics. First, a model based on Fanno flow model is constructed. Second, model is studied in which the flow in the tube is isothermal. The flow in the tube in many cases is somewhere between the Fanno flow model to Isothermal flow model. This reality is an additional reason for the construction of two models in which they can be compared.

Effects such as chemical reactions (or condensation/evaporation) are neglected. There are two suggested itself possibilities to the connection between the tube to the tank (see the Figure 12.2): one) direct two) through a reduction. The direct connection is when the tube is connect straight to tank like in a case where pipe is welded into the tank. The reduction is typical when a ball is filled trough an one-way valve (filling a baseball ball, also in manufacturing processes). The second possibility leads itself to an additional parameter that is independent of the resistance. The first kind connection tied the resistance, $\frac{4fL}{D}$, with the tube area.

The simplest model for gas inside the chamber as a first approximation is the isotropic model. It is assumed that kinetic change in the chamber is negligible. Therefore, the pressure in the chamber is equal to the stagnation pressure, $P \approx P_0$ (see Figure (12.4)). Thus, the stagnation pressure at the tube's entrance is the same as the pressure in the chamber.

The mass in the chamber and mass flow out are expressed in terms of the chamber variables (see Figure 12.3. The mass in the tank for perfect gas reads

$$\frac{dm}{dt} - \dot{m}_{out} = 0 \quad (12.1)$$

And for perfect gas the mass at any given time is

$$m = \frac{P(t)V(t)}{RT(t)} \quad (12.2)$$

The mass flow out is a function of the resistance in tube, $\frac{4fL}{D}$ and the pressure difference between the two sides of the tube $\dot{m}_{out}(\frac{4fL}{D}, P_1/P_2)$. The initial conditions in the chamber are $T(0), P(0)$ and etc. If the mass occupied in the tube is neglected (only for filling process) the most general equation ideal gas (12.1) reads

$$\frac{d}{dt} \left(\overbrace{\frac{PV}{RT}}^m \right) \pm \overbrace{\rho_1 A c_1 M_1 \left(\frac{4fL}{D}, \frac{P_2}{P_1} \right)}^{\dot{m}_{out}} = 0 \quad (12.3)$$

When the plus sign is for filling process and the negative sign is for evacuating process.

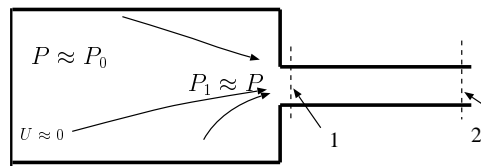


Fig. -12.4. The pressure assumptions in the chamber and tube entrance

12.2 General Model and Non-dimensioned

It is convenient to non-dimensioned the properties in chamber by dividing them by their initial conditions. The dimensionless properties of chamber as

$$\bar{T} = \frac{T(t = \bar{t})}{T(t = 0)} \quad (12.4a)$$

$$\bar{V} = \frac{V(t = \bar{t})}{V(t = 0)} \quad (12.4b)$$

$$\bar{P} = \frac{P(t = \bar{t})}{P(t = 0)} \quad (12.4c)$$

$$\bar{t} = \frac{t}{t_c} \quad (12.4d)$$

where t_c is the characteristic time of the system defined as followed

$$t_c = \frac{V(0)}{AM_{max}\sqrt{kRT(0)}} \quad (12.5)$$

The physical meaning of characteristic time, t_c is the time that will take to evacuate the chamber if the gas in the chamber was in its initial state, the flow rate was at its maximum (choking flow), and the gas was incompressible in the chamber. Utilizing these definitions (12.4) and substituting into equation (12.3) yields

$$\frac{P(0)V(0)}{t_c RT(0)} \frac{d}{d\bar{t}} \left(\frac{\bar{P}\bar{V}}{\bar{T}} \right) \pm \overbrace{\frac{\bar{P}_1}{RT_1} \frac{P(0)}}{\rho} A \overbrace{\sqrt{kRT_1 T(0)} M_{max} \bar{M}(\bar{t})}^{c(t)} = 0 \quad (12.6)$$

where the following definition for the reduced Mach number is added as

$$\bar{M} = \frac{M_1(t)}{M_{max}} \quad (12.7)$$

After some rearranging equation (12.6) obtains the form

$$\frac{d}{d\bar{t}} \left(\frac{\bar{P}\bar{V}}{\bar{T}} \right) \pm \frac{t_c AM_{max} \sqrt{kRT(0)}}{V(0)} \frac{\bar{P}_1 \bar{M}_1}{\sqrt{T_1}} \bar{M} = 0 \quad (12.8)$$

and utilizing the definition of characteristic time, equation (12.5), and substituting into equation (12.8) yields

$$\frac{d}{d\bar{t}} \left(\frac{\bar{P}\bar{V}}{\bar{T}} \right) \pm \frac{\bar{P}_1 \bar{M}}{\sqrt{T_1}} = 0 \quad (12.9)$$

Note that equation (12.9) can be modified by introducing additional parameter which referred to as external time, t_{max} ³. For cases, where the process time is important parameter equation (12.9) transformed to

$$\frac{d}{d\tilde{t}} \left(\frac{\bar{P}\bar{V}}{\bar{T}} \right) \pm \frac{t_{max}}{t_c} \frac{\bar{P}_1\bar{M}}{\sqrt{\bar{T}_1}} = 0 \quad (12.10)$$

when \bar{P} , \bar{V} , \bar{T} , and \bar{M} are all are function of \tilde{t} in this case. And where $\tilde{t} = t/t_{max}$. It is more convenient to deal with the stagnation pressure then the actual pressure at the entrance to the tube. Utilizing the equations developed in Chapter 5 between the stagnation condition, denoted without subscript, and condition in a tube denoted with subscript 1. The ratio of $\frac{\bar{P}_1}{\sqrt{\bar{T}_1}}$ is substituted by

$$\frac{\bar{P}_1}{\sqrt{\bar{T}_1}} = \frac{\bar{P}}{\sqrt{\bar{T}}} \left[1 + \frac{k-1}{2} M^2 \right]^{\frac{-(k+1)}{2(k-1)}} \quad (12.11)$$

It is convenient to denote

$$f[M] = \left[1 + \frac{k-1}{2} M^2 \right]^{\frac{-(k+1)}{2(k-1)}} \quad (12.12)$$

Note that $f[M]$ is a function of the time. Utilizing the definitions (12.11) and substituting equation (12.12) into equation (12.9) to be transformed into

$$\frac{d}{d\tilde{t}} \left(\frac{\bar{P}\bar{V}}{\bar{T}} \right) \pm \frac{\bar{P}\bar{M}(\tilde{t})f[M]}{\sqrt{\bar{T}}} = 0 \quad (12.13)$$

Equation (12.13) is a first order nonlinear differential equation that can be solved for different initial conditions. At this stage, the author isn't aware that there is a general solution for this equation⁴. Nevertheless, many numerical methods are available to solve this equation.

12.2.1 Isentropic Process

The relationship between the pressure and the temperature in the chamber can be approximated as isotropic and therefore

$$\bar{T} = \frac{T(t)}{T(0)} = \left[\frac{P(t)}{P(0)} \right]^{\frac{k-1}{k}} = \bar{P}^{\frac{k-1}{k}} \quad (12.14)$$

³This notation is used in many industrial processes where time of process referred to sometime as the maximum time.

⁴To those mathematically included, find the general solution for this equation.

The ratios can be expressed in term of the reduced pressure as followed:

$$\frac{\bar{P}}{\bar{T}} = \frac{\bar{P}}{\bar{P}^{\frac{k-1}{k}}} = \bar{P}^{\frac{1}{k}} \quad (12.15)$$

and

$$\frac{\bar{P}}{\sqrt{\bar{T}}} = \bar{P}^{\frac{k+1}{2k}} \quad (12.16)$$

So equation (12.13) is simplified into three different forms:

$$\frac{d}{dt} \left(\bar{V} \bar{P}^{\frac{1}{k}} \right) \pm \bar{P}^{\frac{k+1}{2k}} \bar{M}(\bar{t}) f[M] = 0 \quad (12.17a)$$

$$\frac{1}{k} \bar{P}^{\frac{1-k}{k}} \frac{d\bar{P}}{dt} \bar{V} + \bar{P}^{\frac{1}{k}} \frac{d\bar{V}}{dt} \pm \bar{P}^{\frac{k+1}{2k}} \bar{M}(\bar{t}) f[M] = 0 \quad (12.17b)$$

$$\bar{V} \frac{d\bar{P}}{dt} + k \bar{P} \frac{d\bar{V}}{dt} \pm k \bar{P}^{\frac{3k-1}{2k}} \bar{M}(\bar{t}) f[M] = 0 \quad (12.17c)$$

Equation (12.17) is a general equation for evacuating or filling for isentropic process in the chamber. It should be point out that, in this stage, the model in the tube could be either Fanno flow or Isothermal flow. The situations where the chamber undergoes isentropic process but the flow in the tube is Isothermal are limited. Nevertheless, the application of this model provide some kind of a limit where to expect when some heat transfer occurs. Note the temperature in the tube entrance can be above or below the surrounding temperature. Simplified calculations of the entrance Mach number are described in the advance topics section.

12.2.2 Isothermal Process in The Chamber

12.2.3 A Note on the Entrance Mach number

The value of Mach number, M_1 is a function of the resistance, $\frac{4fL}{D}$ and the ratio of pressure in the tank to the back pressure, P_B/P_1 . The exit pressure, P_2 is different from P_B in some situations. As it was shown before, once the flow became choked the Mach number, M_1 is only a function of the resistance, $\frac{4fL}{D}$. These statements are correct for both Fanno flow and the Isothermal flow models. The method outlined in Chapters 9 and 10 is appropriate for solving for entrance Mach number, M_1 .

Two equations must be solved for the Mach numbers at the duct entrance and exit when the flow is in a chokeless condition. These equations are combinations of the momentum and energy equations in terms of the Mach numbers. The characteristic equations for Fanno flow (10.50), are

$$\frac{4fL}{D} = \left[\frac{4fL}{D} \Big|_{max} \right]_1 - \left[\frac{4fL}{D} \Big|_{max} \right]_2 \quad (12.18)$$

and

$$\frac{P_2}{P_0(t)} = \left[1 + \frac{k-1}{2} M_2^2 \right]^{\frac{k}{1-k}} \frac{M_1}{M_2} \sqrt{\left[\frac{1 + \frac{k-1}{2} M_2^2}{1 + \frac{k-1}{2} M_1^2} \right]^{\frac{k+1}{k-1}}} \quad (12.19)$$

where $\frac{4fL}{D}$ is defined by equation (10.49).

The solution of equations (12.18) and (12.19) for given $\frac{4fL}{D}$ and $\frac{P_{exit}}{P_0(t)}$ yields the entrance and exit Mach numbers. See advance topic about approximate solution for large resistance, $\frac{4fL}{D}$ or small entrance Mach number, M_1 .

12.3 Rigid Tank with Nozzle

he most simplest possible combination is discussed here before going trough the more complex cases A chamber is filled or evacuated by a nozzle. The gas in the chamber assumed to go an isentropic processes and flow is bounded in nozzle between isentropic flow and isothermal flow⁵. Here, it also will be assumed that the flow in the nozzle is either adiabatic or isothermal.

12.3.1 Adiabatic Isentropic Nozzle Attached

The mass flow out is given by either by Fliegner's equation (5.46) or simply use $cM\rho A^*$ and equation (12.17) becomes

$$\frac{1}{k} \bar{P}^{\frac{1-k}{k}} \frac{d\bar{P}}{d\bar{t}} \pm \bar{P}^{\frac{k+1}{2k}} (\bar{t}) f[M] = 0 \quad (12.20)$$

It was utilized that $\bar{V} = 1$ and \bar{M} definition is simplified as $\bar{M} = 1$. It can be noticed that the characteristic time defined in equation (12.5) reduced into:

$$t_c = \frac{V(0)}{A\sqrt{kRT(0)}} \quad (12.21)$$

Also it can be noticed that equation (12.12) simplified into

$$f[M] = \left[1 + \frac{k-1}{2} M^2 \right]^{\frac{-(k+1)}{2(k-1)}} = \left(\frac{k+1}{2} \right)^{\frac{-(k+1)}{2(k-1)}} \quad (12.22)$$

Equation (12.20) can be simplified as

$$\frac{1}{k} \left(P^{\frac{1-k}{2k}} \right) dP \pm f[m] d\bar{t} = 0 \quad (12.23)$$

⁵This work is suggested by Donald Katze the point out that this issue appeared in Shapiro's Book Vol 1, Chapter 4, p. 111 as a question 4.31.

Equation (12.23) can be integrated as

$$\int_1^{\bar{P}} P^{\frac{1-k}{2k}} dP \pm \int_0^t dt = 0 \quad (12.24)$$

The integration limits are obtained by simply using the definitions of reduced pressure, at $P(\bar{t} = 0) = 1$ and $P(\bar{t} = \bar{t}) = \bar{P}$. After the integration, equation (12.24) and rearrangement becomes

$$\bar{P} = \left[1 \pm \left(\frac{k-1}{2} \right) f[M]\bar{t} \right]^{\frac{2k}{1-k}} \quad (12.25)$$

Example 12.1:

A chamber is connected to a main line with pressure line with a diaphragm and nozzle. The initial pressure at the chamber is 1.5[Bar] and the volume is 1.0[m³]. Calculate time it requires that the pressure to reach 5[Bar] for two different nozzles throat area of 0.001, and 0.1 [m²] when diaphragm is erupted. Assumed the stagnation temperature at the main line is the ambient of 27[°C].

SOLUTION

The characteristic time is

$$t_{max} = \frac{V}{A^*c} = \frac{V}{A^*c} = \frac{1.0}{0.1\sqrt{1.4 \times 287 \times 300}} = 0.028[sec] \quad (12.26)$$

And for smaller area

$$t_{max} = \frac{1.0}{0.001\sqrt{1.4 \times 287 \times 300}} = 2.8[sec]$$

$$\bar{P} = \frac{P(t)}{P(0)} = \frac{4.5}{1.5} = 3.0$$

The time is

$$t = t_{max} \left[\bar{P}^{\frac{1-k}{k}} - 1 \right] \left(\frac{k+1}{2} \right)^{-0} \quad (12.27)$$

Substituting values into equation (12.27) results

$$t = 0.028 \left[3^{\frac{1-1.4}{2.8}} - 1 \right] \left(\frac{2.4}{2} \right)^{\frac{-2.4}{0.8}} = 0.013[sec] \quad (12.28)$$

Filling/Evacuating The Chamber Under choked Condition

The flow in the nozzle can become unchoked and it can be analytically solved. Owczarek [1964] found an analytical solution which described here.

12.3.2 Isothermal Nozzle Attached

In this case the process in nozzle is assumed to isothermal but the process in the chamber is isentropic. The temperature in the nozzle is changing because the temperature in the chamber is changing. Yet, the differential temperature change in the chamber is slower than the temperature change in nozzle. For rigid volume, $\bar{V} = 1$ and for isothermal nozzle $\bar{T} = 1$ Thus, equation (12.13) is reduced into

$$\frac{d\bar{P}}{d\bar{t}} = \pm f[M]\bar{P} = 0 \quad (12.29)$$

Separating the variables and rearranging equation (12.29) converted into

$$\int_1^{\bar{P}} \frac{d\bar{P}}{\bar{P}} \pm f[M] \int_0^{\bar{t}} d\bar{t} = 0 \quad (12.30)$$

Here, $f[M]$ is expressed by equation (12.22). After the integration, equation (12.30) transformed into

$$\begin{aligned} \ln \bar{P} &= \left(\frac{k+1}{2} \right)^{\frac{-(k+1)}{2(k-1)}} \bar{t} \\ \bar{P} &= e^{\left[\left(\frac{k+1}{2} \right)^{\frac{-(k+1)}{2(k-1)}} \bar{t} \right]} \end{aligned} \quad (12.31)$$

12.4 Rapid evacuating of a rigid tank

12.4.1 With Fanno Flow

The relative Volume, $\bar{V}(t) = 1$, is constant and equal one for a completely rigid tank. In such case, the general equation (12.17) "shrinks" and doesn't contain the relative volume term.

A reasonable model for the tank is isentropic (can be replaced polytropic relationship) and Fanno flow are assumed for the flow in the tube. Thus, the specific governing equation is

$$\frac{d\bar{P}}{d\bar{t}} - k\bar{M}f[M]\bar{P}^{\frac{3k-1}{2k}} = 0 \quad (12.32)$$

For a choked flow the entrance Mach number to the tube is at its maximum, M_{max} and therefore $\bar{M} = 1$. The solution of equation (12.32) is obtained by noticing that \bar{M}

is not a function of time and by variables separation results in

$$\int_0^{\bar{t}} d\bar{t} = \int_1^{\bar{P}} \frac{d\bar{P}}{k\bar{M}f[M]\bar{P}^{\frac{3k-1}{2k}}} = \frac{1}{k\bar{M}f[M]} \int_1^{\bar{P}} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \quad (12.33)$$

direct integration of equation (12.33) results in

$$\bar{t} = \frac{2}{(k-1)\bar{M}f[M]} \left[\bar{P}^{\frac{1-k}{2k}} - 1 \right] \quad (12.34)$$

It has to be realized that this is “reversed” function i.e. \bar{t} is a function of P and can be reversed for case. But for the choiced case it appears as

$$\bar{P} = \left[1 + \frac{(k-1)\bar{M}f[M]}{2} \bar{t} \right]^{\frac{2k}{1-k}} \quad (12.35)$$

The function is drawn as shown here in Figure (12.5). The Figure (12.5) shows

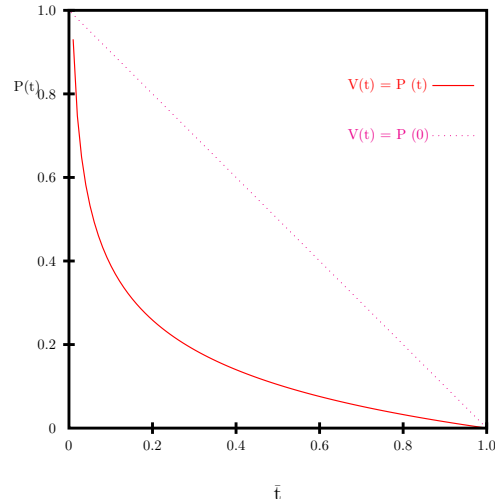


Fig. -12.5. The reduced time as a function of the modified reduced pressure

that when the modified reduced pressure equal to one the reduced time is zero. The reduced time increases with decrease of the pressure in the tank.

At certain point the flow becomes chokeless flow (unless the back pressure is complete vacuum). The transition point is denoted here as chT . Thus, equation (12.34) has to include the entrance Mach under the integration sign as

$$\bar{t} - \bar{t}_{chT} = \int_{P_{chT}}^{\bar{P}} \frac{1}{k\bar{M}f[M]} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \quad (12.36)$$

For practical purposes if the flow is choked for more than 30% of the characteristic time the choking equation can be used for the whole range, unless extra long time or extra low pressure is calculated/needed. Further, when the flow became chokeless the entrance Mach number does not change much from the choking condition.

Again, for the special cases where the choked equation is not applicable the integration has to be separated into zones: choked and chokeless flow regions. And in the choke region the calculations can use the choking formula and numerical calculations for the rest.

Example 12.2:

A chamber with volume of $0.1[m^3]$ is filled with air at pressure of $10[Bar]$. The chamber is connected with a rubber tube with $f = 0.025$, $d = 0.01[m]$ and length of $L = 5.0[m]$

SOLUTION

The first parameter that calculated is $\frac{4fL}{D} \frac{4fL}{D} = 5$

End solution

12.4.2 Filling Process

The governing equation is

$$\frac{d\bar{P}}{d\bar{t}} - k\bar{M}f[M]\bar{P}^{\frac{3k-1}{2k}} = 0 \quad (12.37)$$

For a choked flow the entrance Mach number to the tube is at its maximum, M_{max} and therefore $\bar{M} = 1$. The solution of equation (12.37) is obtained by noticing that \bar{M} is not a function of time and by variable separation results in

$$\int_0^{\bar{t}} d\bar{t} = \int_1^{\bar{P}} \frac{d\bar{P}}{k\bar{M}f[M]\bar{P}^{\frac{3k-1}{2k}}} = \frac{1}{k\bar{M}f[M]} \int_1^{\bar{P}} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \quad (12.38)$$

direct integration of equation (12.38) results in

$$\bar{t} = \frac{2}{(k-1)\bar{M}f[M]} \left[\bar{P}^{\frac{1-k}{2k}} - 1 \right] \quad (12.39)$$

It has to be realized that this is a reversed function. Nevertheless, with today computer this should not be a problem and easily can be drawn as shown here in Figure (12.5). The Figure shows that when the modified reduced pressure equal to one the reduced time is zero. The reduced time increases with decrease of the pressure in the tank.

At some point the flow becomes chokeless flow (unless the back pressure is a complete vacuum). The transition point is denoted here as chT . Thus, equation (12.39) has to include the entrance Mach under the integration sign as

$$\bar{t} - \bar{t}_{chT} = \int_{P_{chT}}^{\bar{P}} \frac{1}{k\bar{M}f[M]} \bar{P}^{\frac{1-3k}{2k}} d\bar{P} \quad (12.40)$$

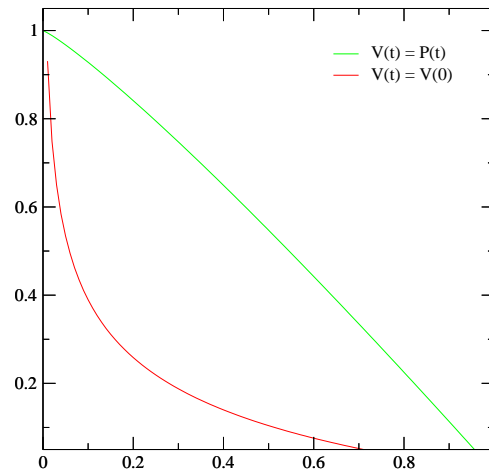


Fig. -12.6. The reduced time as a function of the modified reduced pressure

12.4.3 The Isothermal Process

For Isothermal process, the relative temperature, $\bar{T} = 1$. The combination of the isentropic tank and Isothermal flow in the tube is different from Fanno flow in that the choking condition occurs at $1/\sqrt{k}$. This model is reasonably appropriated when the chamber is insulated and not flat while the tube is relatively long and the process is relatively long.

It has to be remembered that the chamber can undergo isothermal process. For the double isothermal (chamber and tube) the equation (12.6) reduced into

$$\frac{P(0)V(0)}{t_c RT(0)} \frac{d(\bar{P}\bar{V})}{d\bar{t}} \pm \frac{\overbrace{P_1}^{\rho} P(0)}{R T(0)} A \sqrt{\overbrace{kRT(0)}^{c(0)}} M_{max} \bar{M}(\bar{t}) = 0 \quad (12.41)$$

12.4.4 Simple Semi Rigid Chamber

A simple relation of semi rigid chamber when the volume of the chamber is linearly related to the pressure as

$$V(t) = aP(t) \quad (12.42)$$

where a is a constant that represent the physics. This situation occurs at least in small ranges for airbag balloon etc. The physical explanation when it occurs beyond the scope of this book. Nevertheless, a general solution is easily can be obtained similarly to rigid tank. Substituting equation (12.42) into yields

$$\frac{d}{dt} \left(\bar{P}^{\frac{1+k}{k}} \right) - \bar{P}^{\frac{k+1}{2k}} \bar{M}f[M] = 0 \quad (12.43)$$

Carrying differentiation result in

$$\frac{1+k}{k} \bar{P}^{\frac{1}{k}} \frac{d\bar{P}}{dt} - \bar{P}^{\frac{k+1}{2k}} \bar{M}f[M] = 0 \quad (12.44)$$

Similarly as before, the variables are separated as

$$\int_0^{\bar{t}} dt = \frac{k}{1+k} \int_1^{\bar{P}} \frac{\bar{P}^{\frac{k-1}{2k}} d\bar{P}}{\bar{M}f[M]} \quad (12.45)$$

The equation (12.45) integrated to obtain the form

$$\bar{t} = \frac{2k^2}{\bar{M}f[M](3k-1)(1+k)} \left[1 - \bar{P}^{\frac{3k-1}{2k}} \right] \quad (12.46)$$

The physical meaning that the pressure remains larger thorough evacuating process, as results in faster reduction of the gas from the chamber.

12.4.5 The “Simple” General Case

The relationship between the pressure and the volume from the physical point of view must be monotonous. Further, the relation must be also positive, increase of the pressure results in increase of the volume (as results of Hook’s law. After all, in the known situations to this author pressure increase results in volume decrease (at least for ideal gas.).

In this analysis and previous analysis the initial effect of the chamber container inertia is neglected. The analysis is based only on the mass conservation and if unsteady effects are required more terms (physical quantities) have taken into account. Further, it is assumed the ideal gas applied to the gas and this assumption isn’t relaxed here.

Any continuous positive monotonic function can be expressed into a polynomial function. However, as first approximation and simplified approach can be done by a single term with a different power as

$$V(t) = aP^n \quad (12.47)$$

When n can be any positive value including zero, 0. The physical meaning of $n = 0$ is that the tank is rigid. In reality the value of n lays between zero to one. When n is

approaching to zero the chamber is approaches to a rigid tank and vis versa when the $n \rightarrow 1$ the chamber is flexible like a balloon.

There isn't a real critical value to n . Yet, it is convenient for engineers to further study the point where the relationship between the reduced time and the reduced pressure are linear⁶ Value of n above it will Convex and below it concave.

$$\frac{d}{dt} \left(\bar{P}^{\frac{1+nk-k}{k}} \right) - \bar{P}^{\frac{k+1}{2k}} \bar{M}f[M] = 0 \quad (12.48)$$

Notice that when $n = 1$ equation (12.49) reduced to equation (12.43).
After carrying-out differentiation results

$$\frac{1+nk-k}{k} \bar{P}^{\frac{1+nk-2k}{k}} \frac{d\bar{P}}{dt} - \bar{P}^{\frac{k+1}{2k}} \bar{M}f[M] = 0 \quad (12.49)$$

Again, similarly as before, variables are separated and integrated as follows

$$\int_0^{\bar{t}} dt = \frac{1+nk-k}{k} \int_1^{\bar{P}} \frac{\bar{P}^{\frac{1+2nk-5k}{2k}} d\bar{P}}{\bar{M}f[M]} \quad (12.50)$$

Carrying-out the integration for the initial part if exit results in

$$\bar{t} = \frac{2k^2}{\bar{M}f[M](3k-2nk-1)(1+k)} \left[1 - \bar{P}^{\frac{3k-2nk-1}{2k}} \right] \quad (12.51)$$

The linear condition are obtain when

$$3k-2nk-1=1 \longrightarrow n = \frac{3k-2}{2k} \quad (12.52)$$

That is just bellow 1 ($n = 0.785714286$) for $k = 1.4$.

12.5 Advance Topics

The term $\frac{4fL}{D}$ is very large for small values of the entrance Mach number which requires keeping many digits in the calculation. For small values of the Mach numbers, equation (12.18) can be approximated as

$$\frac{4fL}{D} = \frac{1}{k} \frac{M_{exit}^2 - M_{in}^2}{M_{exit}^2 M_{in}^2} \quad (12.53)$$

and equation (12.19) as

$$\frac{P_{exit}}{P_0(t)} = \frac{M_{in}}{M_{exit}}. \quad (12.54)$$

⁶Some suggested this border point as infinite evocation to infinite time for evacuation etc. This undersigned is not aware situation where this indeed play important role. Therefore, it is waited to find such conditions before calling it as critical condition.

The solution of two equations (12.53) and (12.54) yields

$$M_{in} = \sqrt{\frac{1 - \left[\frac{P_{exit}}{P_0(t)}\right]^2}{k \frac{4fL}{D}}}. \quad (12.55)$$

This solution should be used only for $M_{in} < 0.00286$; otherwise equations (12.18) and (12.19) must be solved numerically.

The solution of equation (12.18) and (12.19) is described in "Pressure die casting: a model of vacuum pumping" Bar-Meir, G; Eckert, E R G; Goldstein, R. J. *Journal of Manufacturing Science and Engineering (USA)*. Vol. 118, no. 2, pp. 259-265. May 1996.

CHAPTER 13

Evacuating/Filing Chambers under External Volume Control

This chapter is the second on the section dealing with filling and evacuating chambers. Here the model deals with the case where the volume is controlled by external forces. This kind of model is applicable to many manufacturing processes such as die casting, extraction etc. In general the process of the displacing the gas (in many cases air) with a liquid is a very common process. For example, in die casting process liquid metal is injected to a cavity and after the cooling/solidification period a part is obtained in near the final shape. One can also view the exhaust systems of internal combustion engine in the same manner. In these processes, sometime is vital to obtain a proper evacuation of the gas (air) from the cavity.

13.1 General Model

In this analysis, in order to obtain the essence of the process, some simplified assumptions are made. The simplest model of such process is when a piston is displacing the gas through a long tube. It is assumed that no chemical reaction (or condensation/evaporation) occurs in the piston or the tube ¹. It is further assumed that the process is relatively fast. The last assumption is an appropriate assumption in process such as die casting.

Two extreme possibilities again suggest themselves: rapid and slow processes. The two different connections, direct and through reduced area are combined in this analysis.

¹such reaction are possible and expected to be part of process but they complicate the analysis and do not contribute to understanding the compressibility effects.

13.1.1 Rapid Process

Clearly under the assumption of rapid process the heat transfer can be neglected and Fanno flow can be assumed for the tube. The first approximation isotropic process describe the process inside the cylinder (see Figure (13.1)).

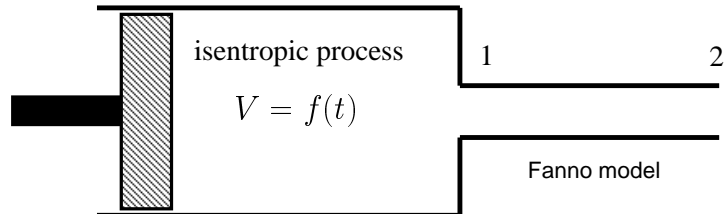


Fig. -13.1. The control volume of the "Cylinder".

Before introducing the steps of the analysis, it is noteworthy to think about the process in qualitative terms. The replacing incompressible liquid enter in the same amount as replaced incompressible liquid. But in a compressible substance the situation can be totally different, it is possible to obtain a situation where that most of the liquid entered the chamber and yet most of the replaced gas can be still be in the chamber. Obtaining conditions where the volume of displacing liquid is equal to the displaced liquid are called the critical conditions. These critical conditions are very significant that they provide guidelines for the design of processes.

Obviously, the best ventilation is achieved with a large tube or area. In manufacture processes to minimize cost and the secondary machining such as trimming and other issues the exit area or tube has to be narrow as possible. In the exhaust system cost of large exhaust valve increase with the size and in addition reduces the strength with the size of valve². For these reasons the optimum size is desired. The conflicting requirements suggest an optimum area, which is also indicated by experimental studies and utilized by practiced engineers.

The purpose of this analysis to yields a formula for critical/optimum vent area in a simple form is one of the objectives of this section. The second objective is to provide a tool to "combine" the actual tube with the resistance in the tube, thus, eliminating the need for calculations of the gas flow in the tube to minimize the numerical calculations.

A linear function is the simplest model that decibels changes the volume. In reality, in some situations like die casting this description is appropriate. Nevertheless, this model can be extended numerical in cases where more complex function is applied.

$$V(t) = V(0) \left[1 - \frac{t}{t_{max}} \right] \quad (13.1)$$

Equation (13.1) can be non-dimensionlased as

$$\bar{V}(\bar{t}) = 1 - \bar{t} \quad (13.2)$$

²After certain sizes, the possibility of crack increases.

The governing equation (12.10) that was developed in the previous Chapter (12) obtained the form as

$$[\bar{P}]^{\frac{1}{k}} \left\{ \frac{1}{k} \frac{\bar{V}}{\bar{P}} \frac{d\bar{P}}{d\bar{t}} + \frac{d\bar{V}}{d\bar{t}} \right\} + \frac{t_{max} \bar{M} f(M)}{t_c} [\bar{P}]^{\frac{k+1}{2k}} = 0 \quad (13.3)$$

where $\bar{t} = t/t_{max}$. Notice that in this case that there are two different characteristic times: the “characteristic” time, t_c and the “maximum” time, t_{max} . The first characteristic time, t_c is associated with the ratio of the volume and the tube characteristics (see equation (12.5)). The second characteristic time, t_{max} is associated with the imposed time on the system (in this case the elapsed time of the piston stroke).

Equation (13.3) is an nonlinear first order differential equation and can be rearranged as follows

$$\frac{d\bar{P}}{k \left(1 - \frac{t_{max}}{t_c} \bar{M} f[M] \bar{P}^{\frac{k-1}{2k}} \right) \bar{P}} = \frac{d\bar{t}}{1 - \bar{t}} \quad ; \quad \bar{P}(0) = 1. \quad (13.4)$$

Equation (13.4) is can be solved only when the flow is choked In which case $f[m]$ isn't function of the time.

The solution of equation (13.4) can be obtained by transforming and by introducing a new variable $\xi = \bar{P}^{\frac{k-1}{2k}}$ and therefore $\bar{P} = [\xi]^{\frac{2k}{k-1}}$. The reduced Pressure derivative, $d\bar{P} = \frac{2k}{k-1} [\xi]^{\left(\frac{2k}{k-1}\right)-1} d\xi$ Utilizing this definition and there implication reduce equation (13.4)

$$\frac{2 [\xi]^{\left(\frac{2k}{k-1}\right)-1} d\xi}{(k-1) (1 - B\xi) [\xi]^{\frac{2k}{k-1}}} = \frac{d\bar{t}}{1 - \bar{t}} \quad (13.5)$$

where $B = \frac{t_{max}}{t_c} \bar{M} f[M]$ And equation (13.5) can be further simplified as

$$\frac{2d\xi}{(k-1) (1 - B\xi) \xi} = \frac{d\bar{t}}{1 - \bar{t}} \quad (13.6)$$

Equation (13.6) can be integrated to obtain

$$\frac{2}{(k-1)B} \ln \left| \frac{1 - B\xi}{\xi} \right| = - \ln \bar{t} \quad (13.7)$$

or in a different form

$$\left| \frac{1 - B\xi}{\xi} \right|^{\frac{2}{(1-k)B}} = \bar{t} \quad (13.8)$$

Now substituting to the “preferred” variable

$$\left[\frac{1 - \frac{t_{max}}{t_c} \bar{M} f[M] \bar{P}^{\frac{k-1}{2k}}}{\bar{P}^{\frac{k-1}{2k}}} \right]^{\frac{2}{(1-k) \frac{t_{max}}{t_c} \bar{M} f[M]}} \Big|_{\bar{P}}^1 = \bar{t} \quad (13.9)$$

The analytical solution is applicable only in the case which the flow is choked thorough all the process. The solution is applicable to indirect connection. This happen when vacuum is applied outside the tube (a technique used in die casting and injection molding to improve quality by reducing porosity.). In case when the flow chokeless a numerical integration needed to be performed. In the literature, to create a direct function equation (13.4) is transformed into

$$\frac{d\bar{P}}{d\bar{t}} = \frac{k \left(1 - \frac{t_{max}}{t_c} \bar{M} f[M] \bar{P}^{\frac{k-1}{2k}} \right)}{1 - \bar{t}} \quad (13.10)$$

with the initial condition of

$$P(0) = 1 \quad (13.11)$$

The analytical solution also can be approximated by a simpler equation as

$$\bar{P} = [1 - \bar{t}]^{\frac{t_{max}}{t_c}} \quad (13.12)$$

The results for numerical evaluation in the case when cylinder is initially at an atmospheric pressure and outside tube is also at atmospheric pressure are presented in Figure (13.2). In this case only some part of the flow is choked (the later part). The results of a choked case are presented in Figure (13.3) in which outside tube condition is in vacuum. These Figures (13.2) and 13.3 demonstrate the importance of the ratio of $\frac{t_{max}}{t_c}$. When $\frac{t_{max}}{t_c} > 1$ the pressure increases significantly and verse versa. Thus, the question remains how the time ratio can be transferred to parameters that can the engineer can design in the system.

Denoting the area that creates the ratio $\frac{t_{max}}{t_c} = 1$ as the critical area, A_c provides the needed tool. Thus the exit area, A can be expressed as

$$A = \frac{A}{A_c} A_c \quad (13.13)$$

The actual times ratio $\frac{t_{max}}{t_c} \Big|_{@A}$ can be expressed as

$$\frac{t_{max}}{t_c} \Big|_{@A} = \frac{t_{max}}{t_c} \Big|_{@A} \overbrace{\frac{t_{max}}{t_c} \Big|_{@A_c}}^1 \quad (13.14)$$

According to equation (12.5) t_c is inversely proportional to area, $t_c \propto 1/A$. Thus, equation (13.14) the t_{max} is canceled and reduced into

$$\left. \frac{t_{max}}{t_c} \right|_{@A} = \frac{A}{A_c} \quad (13.15)$$

Parameters influencing the process are the area ratio, $\frac{A}{A_c}$, and the friction parameter, $\frac{4fL}{D}$. From other detailed calculations the author's thesis (later to be published at this site: www.potto.org). it was found that the influence of the parameter $\frac{4fL}{D}$ on the pressure development in the cylinder is quite small. The influence is small on the residual air mass in the cylinder but larger on the Mach number, M_{exit} . The effects of the area ratio, $\frac{A}{A_c}$, are studied here since it is the dominant parameter.

It is important to point out the significance of the $\frac{t_{max}}{t_c}$. This parameter represents the ratio between the filling time and the evacuating time, the time which would be required to evacuate the cylinder for constant mass flow rate at the maximum Mach number when the gas temperature and pressure remain in their initial values. This parameter also represents the dimensionless area, $\frac{A}{A_c}$, according to the following equation

Figure (13.4) describes the pressure as a function of the dimensionless time for various values of $\frac{A}{A_c}$. The line that represents $\frac{A}{A_c} = 1$ is almost straight. For large values of $\frac{A}{A_c}$ the pressure increases the volume flow rate of the air until a quasi steady state is reached. This quasi steady state is achieved when the volumetric air flow rate out is equal to the volume pushed by the piston. The pressure and the mass flow rate are maintained constant after this state is reached. The pressure in this quasi steady state is a function of $\frac{A}{A_c}$. For small values of $\frac{A}{A_c}$ there is no steady state stage. When $\frac{A}{A_c}$ is greater than one the pressure is concave upward and when $\frac{A}{A_c}$ is less than one the pressure is concave downward as shown in Figures (13.4), which was obtained by an integration of equation (13.9).

13.1.2 Examples

Example 13.1:

Calculate the minimum required vent area for die casting process when the die volume is $0.001[m^3]$ and $\frac{4fL}{D} = 20$. The required solidification time, $t_{max} = 0.03[sec]$.

SOLUTION

End solution

13.1.3 Direct Connection

In the above analysis is applicable to indirect connection. It should be noted that critical area, A_c , is not function of the time. The direct connection posts more mathematical difficulty because the critical area is not constant and time dependent.

To continue

13.2 Summary

The analysis indicates there is a critical vent area below which the ventilation is poor and above which the resistance to air flow is minimal. This critical area depends on the geometry and the filling time. The critical area also provides a mean to “combine” the actual vent area with the vent resistance for numerical simulations of the cavity filling, taking into account the compressibility of the gas flow.

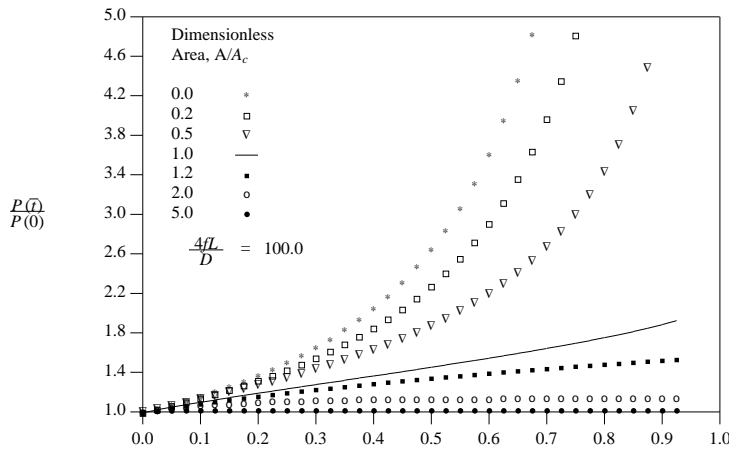


Figure a

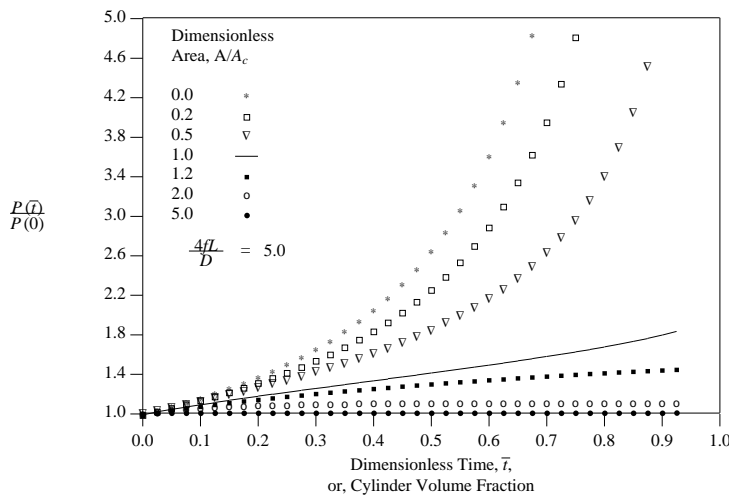


Figure b

Fig. -13.2. The pressure ratio as a function of the dimensionless time for chokeless condition

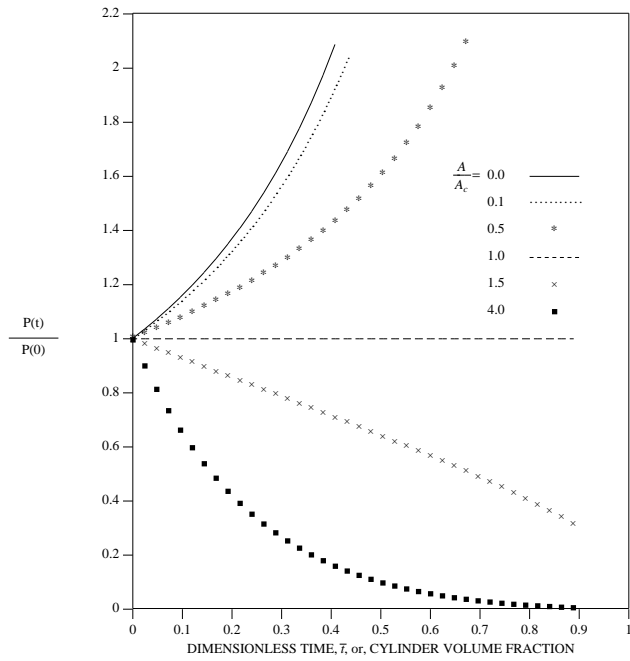


Fig. -13.3. The pressure ratio as a function of the dimensionless time for choked condition

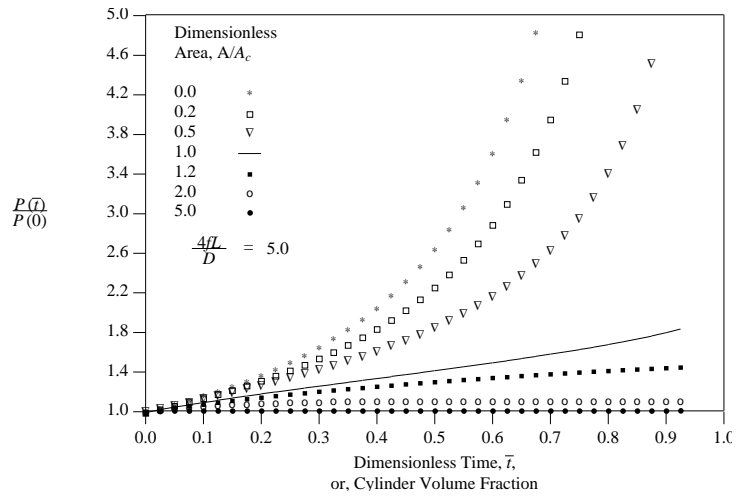


Fig. -13.4. The pressure ratio as a function of the dimensionless time